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Chiang Mai University, Chiang Mai, Thailand

Some properties of S_β-open mappings

Ardoon Jongrak
Mathematics Promgram, Faculty of Science and Technology
Phechabun Rajabhat University, Phechabun 67000, Thailand

Abstract

Khalaf and Ahmed defined S_β–open sets in topological spaces and used it to define and study the concept of S_β–continuous functions in 2013. So, in this study we used S_β–open sets to define S_β–open maps and to study their properties.

Keywords: semi-open sets, β-closed sets, S_β–open sets, S_β–neighborhood, S_β–continuous, S_β–open mappings.

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1 Introduction

Open mappings play an important role in the study of modern mathematics, especially in Topology and Functional analysis. Mathematicians have always defined new types of open maps. In 1969 N. Biswas [1] defined and studied the notions of semi-open maps in terms of semi-open sets. In 1983 A.S. Mashhour, I.A. Hasaneie and S.E. EL-Dee [2] covered and studied the notions of α-open maps in terms of α-open sets. In 1998 G.B. Navalagi [3] discovered the notions of Quasi α-open maps in terms of α-open sets, meanwhile J. Dontchev [4] collected the definitions of several open maps which were defined in terms of pre-open sets. In 2011 S. Balasbranaemian [5] found out the notions of vg-open maps in terms of vg-open sets. In 2013 A.B. Khalaf and N.K. Ahmed [6] introduced a new class of semi-open sets called S_β-open sets, they then examined S_β-continuous functions. The findings from those works lead to this paper with the aims to define S_β-open maps and to study their properties. Preliminaries and some characterizations of S_β-open maps are also introduced in this study.

2 Preliminaries

Thoughout this paper spaces (X,τ) and (Y,σ) (or simply X and Y) always denote topological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of X.The notation of the closure of A and the interior of A are given as $cl(A)$ and $\text{int}(A)$ respectively.

Definition 2.1 Let (X,τ) be a space and $A\subseteq X$. Then A is said to be

(a) semi-open ([7]) if $A\subseteq cl(\text{int}(A))$

(b) β-open ([8]) if $A\subseteq cl(\text{int}(cl(A)))$ and

(c) a semi-open subset A is said to be S$_\beta$-open ([6]) if for each $x \in A$ there exists a β-closed set F such that $x \in F \subseteq A$. (A set is called β-closed if its complement is β-open.)

The complement of a semi-open (resp., β-open, S$_\beta$-open) set is said to be semi-closed (resp., β-closed, S$_\beta$-closed). The union of all semi-open (resp., β-open, S$_\beta$-open) sets of X contained in A is called the semi-interior (resp., β- interior, S$_\beta$- interior) of A denoted by $\operatorname{sint}(A)$ (resp., $\beta\operatorname{int}(A)$, $S_\beta\operatorname{int}(A)$). The intersection of all semi-closed (resp., β-closed, S$_\beta$- closed) sets of X containing a subset A is called the semi-closure (resp., β-closure, S$_\beta$- closure) of A denoted by $scl(A)$ (resp., $\beta\operatorname{cl}(A)$, $S_\beta\operatorname{cl}(A)$). The family of all semi-open (resp., β-open, S$_\beta$-open, semi-closed, β-closed, S$_\beta$- closed) subsets of a topological space X is denoted by $SO(X)$ (resp., $\beta O(X)$, $S_\beta O(X)$, $SC(X)$, $\beta C(X)$, $S_\beta C(X)$).

Definition 2.2 ([6]) Let (X,τ) be a space and $N \subseteq X$ is called an S$_\beta$-neighborhood of a subset A of X, if there exists an S$_\beta$-open set U such that $A \subseteq U \subseteq N$. When $A = \{x\}$ we say that N is an S$_\beta$-neighborhood of x.

Proposition 2.1 ([6]) For any subset A and B of a topological space X, the following statements are true:

1. A is S$_\beta$-open if and only if $A = S_\beta \operatorname{int}(A)$.
2. $\phi = S_\beta \operatorname{int}(\phi)$ and $X = S_\beta \operatorname{int}(X)$.
3. $S_\beta \operatorname{int}(A) \subseteq A$.
4. If $A \subseteq B$ then $S_\beta \operatorname{int}(A) \subseteq S_\beta \operatorname{int}(B)$.
5. $S_\beta \operatorname{int}(A) \cup S_\beta \operatorname{int}(B) \subseteq S_\beta \operatorname{int}(A \cup B)$.
6. $S_\beta \operatorname{int}(A \cap B) \subseteq S_\beta \operatorname{int}(A) \cap S_\beta \operatorname{int}(B)$.
7. $S_\beta \operatorname{int}(A \setminus B) \subseteq S_\beta \operatorname{int}(A) \setminus S_\beta \operatorname{int}(B)$.

Proposition 2.2 ([6]) For any subsets F and E of a topological space X, the following statements are true:

1. F is S$_\beta$-closed set if and only if $F = S_\beta cl(F)$.
2. $\phi = S_\beta cl(\phi)$ and $X = S_\beta cl(X)$.
3. $F \subseteq S_\beta cl(F)$.
4. If $F \subseteq E$ then $S_\beta cl(F) \subseteq S_\beta cl(E)$.
5. $S_\beta cl(F) \cup S_\beta cl(E) \subseteq S_\beta cl(F \cup E)$.
6. $S_\beta cl(F \cap E) \subseteq S_\beta cl(F) \cap S_\beta cl(E)$.
7. $S_\beta cl(X \setminus F) = X \setminus S_\beta \operatorname{int}(F)$ and $X \setminus S_\beta cl(F) = S_\beta \operatorname{int}(X \setminus F)$.

Definition 2.3 ([6]) A function $f:(X,\tau) \to (Y,\sigma)$ is said to be an S$_\beta$-continuous at a point x in X, if for each open set V of Y containing $f(x)$ there exists an S$_\beta$-open set U in X containing x such that $f(U) \subseteq V$. When f is S$_\beta$-continuous at every point x of X we say that f is S$_\beta$-continuous on X.

Theorem 2.1 ([6]) A function $f:(X,\tau) \to (Y,\sigma)$ is S$_\beta$-continuous if and only if the inverse image of every open set in Y is S$_\beta$-open in X.

3 Characterizations

In [3] Quasi α-open mappings are introduced using α-open sets. Similary in this section S_β-open mappings are defined in terms of S_β-open sets and their properties are studied.

Definition 3.1 A mapping $f:(X,\tau)\to(Y,\sigma)$ is said to be S_β-open if $f(U)$ is open in Y for each S_β-open set U in X.

Example 3.2 Let $X=\{a,b,c,d\}$, $\tau=\{\phi,X,\{a\},\{b\},\{a,b\},\{a,b,c\}\}$ be a topology on X, $Y=\{p,q,r\}$, $\sigma=\{\phi,Y,\{q\},\{r\},\{q,r\}\}$ be a topology on Y and $f:X\to Y$ such that $f(a)=r$, $f(b)=q$ and $f(c)=f(d)=r$. Show that f is S_β-open.

Solution. For (X,τ) we get that $S_\beta O(X)=\{\phi,\{b\},\{a,c,d\},X\}$ and $f(\phi)=\phi\in\sigma$, $f(\{b\})=\{q\}\in\sigma$, $f(\{a,c,d\})=Y\in\sigma$ and $f(X)=Y\in\sigma$.
Therefore, f is S_β-open.

Observe that S_β-openness and S_β-continuity are independent. By the above example 3.2 f is not S_β-continuous on X because $f^{-1}(\{r\})=\{c,d\}\notin S_\beta O(X)$ and in example 3.3 f is S_β-continuous but not S_β-open.

Example 3.3 (Ex.4.7 in[6]) Let $X=\{a,b,c,d\},\tau=\{\phi,X,\{a\},\{b,c\},\{a,b,c\}\}$ be a topology on X, then $S_\beta O(X)=\{\phi,\{a\},\{b,c\},\{a,d\},\{a,b,c\},\{b,c,d\},X\}$. The identity function is S_β-continuous which is not S_β-open because $f(\{a,d\})=\{a,d\}\notin\tau$.

Theorem 3.1 A mapping $f:(X,\tau)\to(Y,\sigma)$ is said to be S_β-open if and only if for every subset U of X, $f(S_\beta\operatorname{int}(U))\subseteq\operatorname{int}(f(U))$.

Proof. Let f be an S_β-open map. Now, we have $S_\beta\operatorname{int}(U)\subseteq U$ and $S_\beta\operatorname{int}(U)$ is an S_β-open set. Hence we obtain that $f(S_\beta\operatorname{int}(U))\subseteq f(U)$. Since f be an S_β-open map, that is $f(U)\in\sigma$ and thus $f(S_\beta\operatorname{int}(U))\subseteq\operatorname{int}(f(U))$.
Conversely, assume that conditional holds. If U be an S_β-open set in X.
Then $f(U)=f(S_\beta\operatorname{int}(U))\subseteq\operatorname{int}(f(U))$, but $\operatorname{int}(f(U))\subseteq f(U)$.
Consequently, $f(U)=\operatorname{int}(f(U)$ and hence f is S_β-open.

Theorem 3.2 If a mapping $f:(X,\tau)\to(Y,\sigma)$ is S_β-open then for every subset G of Y, $S_\beta\operatorname{int}(f^{-1}(G))\subseteq f^{-1}(\operatorname{int}(G))$.

Proof. Let G be any arbitrary subset of Y.
Then $S_\beta\operatorname{int}(f^{-1}(G))$ is an S_β-open set in X and f is S_β-open,
then $f(S_\beta\operatorname{int}(f^{-1}(G)))\subseteq\operatorname{int}(f(f^{-1}(G)))\subseteq\operatorname{int}(G)$.
So that $S_\beta\operatorname{int}(f^{-1}(G))\subseteq f^{-1}f(S_\beta\operatorname{int}(f^{-1}(G)))\subseteq f^{-1}(\operatorname{int}(f(f^{-1}(G))))\subseteq f^{-1}(\operatorname{int}(G))$.
Thus $S_\beta\operatorname{int}(f^{-1}(G))\subseteq f^{-1}(\operatorname{int}(G))$.

The converse of Theorem 3.2 may not be true, as the example 4.4 in [6] is showed.

Theorem 3.3 Let $f:X\to Y$ be a mapping. Then the following are equivalent.

(1) f is S_β-open.
(2) For each subset U of X, $f(S_\beta\operatorname{int}(U))\subseteq\operatorname{int}(f(U))$.
(3) For each x in X and each S_β-neighborhood U of x in X there exists a neighborhood V of $f(x)$ in Y such that $V\subseteq f(U)$.

Proof. $(1)\Rightarrow(2)$: It follows from Theorem 3.1.
$(2)\Rightarrow(3)$: Assume (2) holds.

Let $x \in X$ and U be an arbitrary S$_\beta$-neighborhood of x in X.
Then there exists an S$_\beta$-open set V in X such that $x \in V \subseteq U$.
By (2), we have $f(V) = f(S_\beta \operatorname{int}(V)) \subseteq \operatorname{int}(f(V))$ and hence $f(V) = \operatorname{int}(f(V))$.
Therefore it follows that $f(V)$ is open in Y such that $f(x) \in f(V) \subseteq f(U)$.
Thus (3) holds.
$(3) \Rightarrow (1)$: Assume (3) holds. Let U be an arbitrary S$_\beta$-neighborhood in X.
Then for each $y \in f(U)$, by (3) there exists a neighborhood V_y of y in Y such that $V_y \subseteq f(U)$.
As V_y is a neighborhood of y, there exists an open set W_y in Y such that $y \in W_y \subseteq V_y$.
Thus, $f(U) = \cup\{W_y \mid y \in f(U)\}$ is an open set in Y. This implies that f is S$_\beta$-open.

Theorem 3.4 A mapping $f : X \to Y$ is S$_\beta$-open if and only if for any subset B of Y and for any S$_\beta$-closed set F of X containing $f^{-1}(B)$, there exists a closed set A in Y containing B such that $f^{-1}(A) \subseteq F$.
Proof. Suppose f is S$_\beta$-open. Let $B \subseteq Y$ and F be an S$_\beta$-closed set in X containing $f^{-1}(B)$.
Now, put $A = Y - f(X - F)$. It is clear that $f^{-1}(B) \subseteq F$ implies $B \subseteq A$.
Since f is S$_\beta$-open, we obtain that A is a closed set of Y, then we have $f^{-1}(A) \subseteq F$.

Conversely, let U be an S$_\beta$-open set in X and $B = Y - f(U)$.
Then $X - U$ is an S$_\beta$-closed set in X containing $f^{-1}(B)$.
By hypothesis, there exists a closed set F of Y such that $B \subseteq F$ and $f^{-1}(F) \subseteq X - U$.
Hence, we obtain $f(U) \subseteq Y - F$.
On the other hand, it follows that $B \subseteq F$, $Y - F \subseteq Y - B = f(U)$.
Thus, we obtain $f(U) = Y - F$ which is an open set and hence f is S$_\beta$-open map.

Corollary 3.5 A mapping $f : X \to Y$ is S$_\beta$-open if and only if $f^{-1}(cl(B)) \subseteq S_\beta cl(f^{-1}(B))$ for every subset B of Y.
Proof. Subpose that f is S$_\beta$-open. For every subset B of Y, $f^{-1}(B) \subseteq S_\beta cl(f^{-1}(B))$.
Therefore by above Theorem 3.4, there exists a closed set F in Y containing B such that $f^{-1}(F) \subseteq S_\beta cl(f^{-1}(B))$. Because $cl(B) \subseteq cl(F) = F$, we have that $f^{-1}(cl(B)) \subseteq f^{-1}(F)$.
Therefore, we obtain $f^{-1}(cl(B)) \subseteq S_\beta cl(f^{-1}(B))$.

Conversely, let $B \subseteq Y$ and F be an S$_\beta$-closed set of X containing $f^{-1}(B)$.
We have that $S_\beta cl(f^{-1}(B)) \subseteq S_\beta cl(F) = F$.
Put $W = cl_Y(B)$, then we have $B \subseteq cl_Y(B) = W$ and W is closed in Y and by hypothesis, $f^{-1}(W) = f^{-1}(cl(B)) \subseteq S_\beta cl(f^{-1}(B))$, so that $f^{-1}(W) \subseteq F$.
Then by Theorem 3.4 the function f is S$_\beta$-open.

Remark 3.6 The findings in this paper are based for further studies on:

(1) the relations between S$_\beta$-open maps and other types of open maps.
(2) necessary and sufficient conditions for a composition to be S$_\beta$-open maps.
(3) the concept of S$_\beta$-closed maps and S$_\beta$-homeomorphisms in terms of S$_\beta$-open sets.

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