Some properties of $3-$uniform hypergraphs of order n and size at least $\left\lfloor \frac{n}{2} \right\rfloor$

สมบัติบางประการของ $3-$ไฮเพอร์กราฟเอกรูป ที่มีอันดับ n และขนาดอย่างน้อย $\left\lfloor \frac{n}{2} \right\rfloor$

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Abstract

The hypergraph $H=(V,E)$ is said to be order n and size m, if $|V|=n$ and $|E|=m$, respectively and the hypergraph H is said to be $k-$uniform if every edge $e\in E$, $|e|=k$. In this paper, some properties of $k-$uniform hypergraphs of order n and size at least $\left\lfloor \frac{n}{2} \right\rfloor$ are investigated.

Keywords: 3-uniform hypergraph / connected hypergraph

บทคัดย่อ

ให้ $H=(V,E)$ เป็นไฮเพอร์กราฟ จะกล่าวว่า H มีอันดับ n และขนาด m ถ้า $|V|=n$ และ $|E|=m$ ตามลำดับ และจะเรียกไฮเพอร์กราฟ H ว่า $k-$เอกรูป ถ้า $|e|=k$ ทุก $e\in E$ ในบทความนี้ ศึกษาสมบัติบางประการของ $k-$ไฮเพอร์กราฟเอกรูป อันดับ n และ ขนาดอย่างน้อย $\left\lfloor \frac{n}{2} \right\rfloor$

คำสำคัญ: ไฮเพอร์กราฟ / 3-ไฮเพอร์กราฟเอกรูป

1. Preliminaries

A hypergraph H is a pair $H=(X,\ E)$ where X is a finite nonempty set and E is a nonempty subset of power set of X. The elements of X are called vertices of H and those of E edges of H. Here we investigate the domination problems in hypergraphs which all edges have same cardinality called uniform hypergraphs.

A $k-$edge in hypergraph H is an edge of size k. The hypergraph H is said to be $k-$uniform hypergraph if every edge of H is a $k-$edge. Properties of uniform hypergraphs has been studied in several papers (see, e.g. (Clemens, 2015, Eustis et al., 2016, Frank et al., 2003, Han & Zhao, 2015 and Mycroft, 2016)).

We shall use the notation $n_H = |V|$ and $m_H = |E|$, and sometimes simply n and m without subscript if actual H need not be emphasized, to denote the order and size of H, respectively. The edge of set E is often allowed to be a multiset in the literature, but in this research, we exclude multiple edges.

An *isolated vertex* in H is a vertex in H that does not contain in any other edges in H. An *isolated edge* in H is an edge on H that does not intersect any other edge in H. Also in the problems studied here, one may assume that $|e| \geq 2$ holds for all $e \in E$ and H has no isolated vertex.

The degree of a vertex v in H, denoted by $d_H(v)$ or $d(v)$ if H is clear from the context, is the number of edges of H which contain v. A vertex of degree t is called a *degree $t-$ vertex*. The number of degree $t-$ vertices in H is denoted by $n_t(H)$. The minimum degree among the vertices of H is denoted by $\delta(H)$ and the maximum degree by $\Delta(H)$.

2. Characterization of $3-$uniform hypergraphs

In this section, we give some useful properties of 3-uniform hypergraph. Recall that, $3-$uniform hypergraph $H=(V,E)$ is a hypergraph which every edge of H is an $3-$edge, i.e. $|e|=3$ for all $e \in E$. The following definition denote $\mathbb{R}$ and $\mathbf{Z}$ are sets of real numbers and integers, respectively.

The *floor* and *ceiling* functions are map from a real number to the greatest preceding or the least succeeding integer, respectively. More precisely, let $x \in \mathbb{R}$ and $m, n \in \mathbf{Z}$, floor $(\lfloor x \rfloor)$ and ceiling $(\lceil x \rceil)$ may be defined by the set equations,

$$\lfloor x \rfloor = max\{m \in \mathbf{Z} \mid m \leq x\}$$
$$\lceil x \rceil = min\{n \in \mathbf{Z} \mid n \geq x\}$$

Observation 2.1 Let $H=(V,E)$ be a $3-$uniform hypergraph of size m, then $\sum_{v \in V} d(v) = 3m.$

For a hypergraph $H=(V,E)$ of size m, we have $\Delta(H) \leq m$ so, we let $V_i = \{x \in V \mid d(x) \geq i\}$ for $i \in \{1,2,3\ldots,m\}$.

Example 2.2 Let $H=(V,E)$ be a hypergraph as follow:

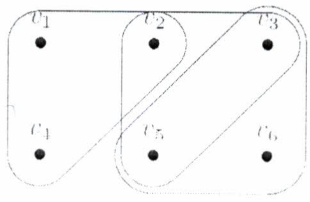

Figure 1: $H = (V, E)$

and from definitions of V_i we have

i) $V_1 = \{x \in V \mid d(x) \geq 1\} = \{v_1, v_2, v_3, v_4, v_5, v_6\} = V$

ii) $V_2 = \{x \in V \mid d(x) \geq 2\} = \{v_2, v_3, v_5\}$

iii) $|V_1| = |V|$ and $|V_1| + |V_2| = 9 = 3(3)$.

Form the above example, we get the following lemma.

Lemma 2.3 Let $H = (V, E)$ be a hypergraph of order n, size m and $\delta(H) \geq 1$. Then $\sum_{v \in V} d(v) = \sum_{i=1}^{m} |V_i| = n + \sum_{i=2}^{m} |V_i|$.

Proof. Let $V(H) = \{v_1, v_2, \mathrm{K}, v_n\}$ such that $d(v_1) = k_1, d(v_2) = k_2, \mathrm{K}, d(v_n) = k_n$. Form $\delta(H) \geq 1$ we have $|V_1| = n$ and $1 \leq k_i \leq m$ for all i since $|E(H)| = m.$.

Thus $\sum_{i=1}^{n} d(v_i) = k_1 + k_2 + \mathrm{K} + k_n$. We will show that $\sum_{i=1}^{m} |V_i| = k_1 + k_2 + \mathrm{K} + k_n$.

Assume that $1 \leq k_1 \leq k_2 \leq \mathrm{K} \leq k_n \leq m$.

So $$\sum_{i=1}^{m} |V_i| = \sum_{i=1}^{k_1} |V_i| + \sum_{i=k_1+1}^{k_2} |V_i| + \mathrm{K} + \sum_{i=k_{n-1}+1}^{k_n} |V_i|, \quad (\text{if } k_n < m \text{ , } V_j = \varnothing \text{ for all } k < j \leq m)$$

$$= k_1 n + (k_2 - k_1)(n-1) + (k_3 - k_2)(n-2) + \mathrm{K} + (k_n - k_{n-1})(n-(n-1))$$

$$= k_1 + k_2 + \mathrm{K}\ k_n$$

Therefore $\sum_{v \in V} d(v) = \sum_{i=1}^{m} |V_i| = n + \sum_{i=2}^{m} |V_i|$.

Lemma 2.4 *Let* $H = (V, E)$ *be a* $3-$*uniform hypergraph of order* n *and size* m.

Then $\sum_{i=2}^{m}|V_i| \geq m-1$.

Proof. Consider $\sum_{v\in V} d(v) = \sum_{i=1}^{m}|V_i| = n + \sum_{i=2}^{m}|V_i|$, we have

$$3m = n + \sum_{i=2}^{m}|V_i|$$

$$m = \frac{n}{3} + \frac{\sum_{i=2}^{m}|V_i|}{3}$$

$$m - \frac{n}{3} = \frac{\sum_{i=2}^{m}|V_i|}{3} \leq \sum_{i=2}^{m}|V_i|$$

Form $n \geq 3$, we conclude that $\sum_{i=2}^{m}|V_i| \geq m-1$ □

Theorem 2.5 *Let* $H=(V,E)$ *be a* $3-$*uniform hypergraph of order* n *and size* m. *Then*

1. *For* n *is odd,* $m \geq \left\lfloor \frac{n}{2} \right\rfloor$ *if and only if* $\sum_{v\in V} d(v) \geq n+m-1$.

2. *For* n *is even,* $m \geq \frac{n}{2}-1$ *if and only if* $\sum_{v\in V} d(v) \geq n+m-2$.

Proof.

1. Let $n = 2q+1$ for some $q \in \mathbf{Z}$, from assumption we have,

$$\begin{aligned} m &\geq \left\lfloor \frac{2q+1}{2} \right\rfloor \\ &= \left\lfloor q + \frac{1}{2} \right\rfloor \\ &= q \end{aligned}$$

This implies that $m \geq q$ Suppose that $\sum_{v\in V} d(v) < n+m-1$. Since $\sum_{v\in V} d(v) = 3m$, $3m < n+m-1 = 2q+1+m-1$. Then $2m < 2q$. This mean that $m < q$, it is impossible. Thus $\sum_{v\in V} d(v) \geq n+m-1$.

Conversely, let $\sum_{v\in V} d(v) \geq n+m-1$.

We have

$$\sum_{v\in V} d(v) = 3m \geq n+m-1$$
$$2m \geq n-1$$
$$m \geq \frac{n-1}{2} \geq \left\lfloor \frac{n-1}{2} \right\rfloor$$

if $n = 2q+1$, then

$$m \geq \left\lfloor \frac{n-1}{2} \right\rfloor = \left\lfloor \frac{2q+1-1}{2} \right\rfloor = q = \left\lfloor \frac{2q}{2} \right\rfloor = \left\lfloor \frac{2q}{2} + \frac{1}{2} \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor$$

Therefore $m \geq \left\lfloor \frac{n}{2} \right\rfloor$

2. Let $n = 2q$ for some $q \in \mathbf{Z}$, from assumption we have,

$$m \geq \frac{2q}{2} - 1 = q-1$$

This implies that $m \geq q-1$. Suppose that $\sum_{v\in V} d(v) < n+m-2$. Since $\sum_{v\in V} d(v) = 3m$, $3m < n+m-2 = 2q+m-2$. Then $2m < 2q-2$. This mean that $m < q-1$, it is impossible. Thus $\sum_{v\in V} d(v) \geq n+m-2$.

Conversely, let $\sum_{v\in V} d(v) \geq n+m-2$.

We have

$$\sum_{v\in V} d(v) = 3m \geq n+m-2$$
$$2m \geq n-2$$
$$m \geq \frac{n-2}{2} = \frac{n}{2} - 1$$

Therefore $m \geq \frac{n}{2} - 1$

□

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