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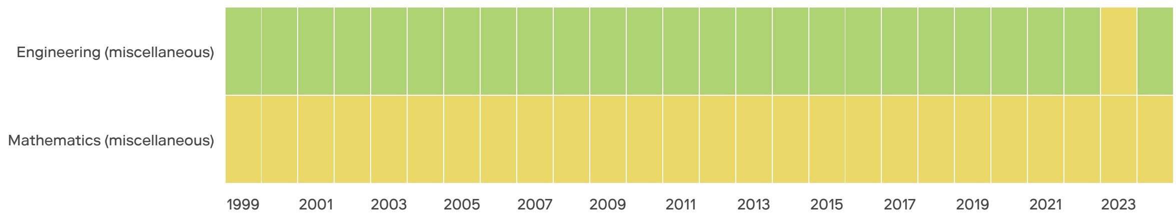

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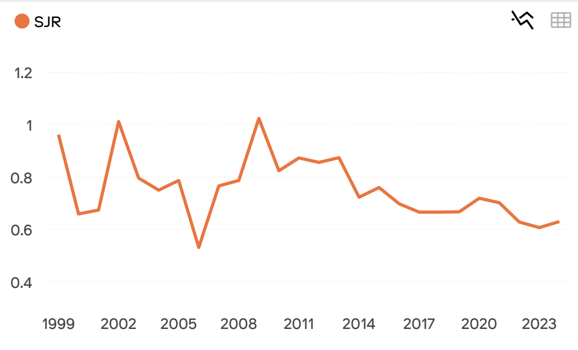

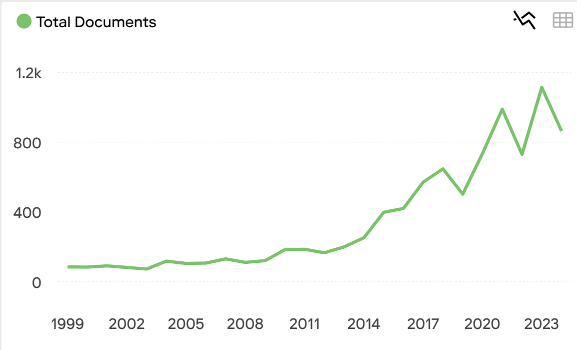

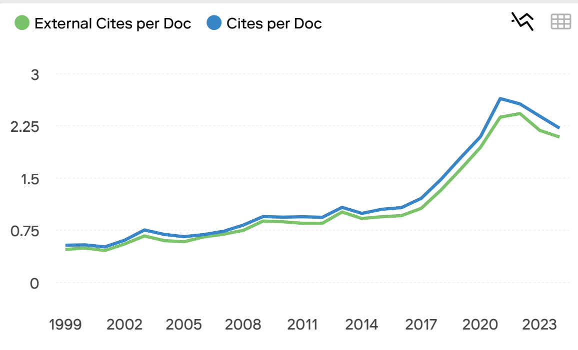

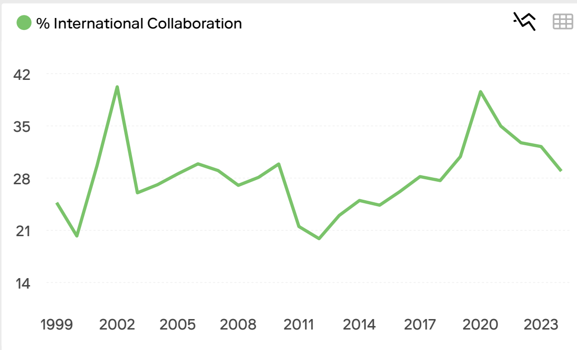

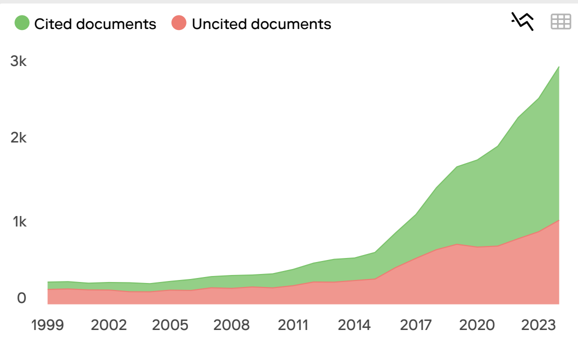

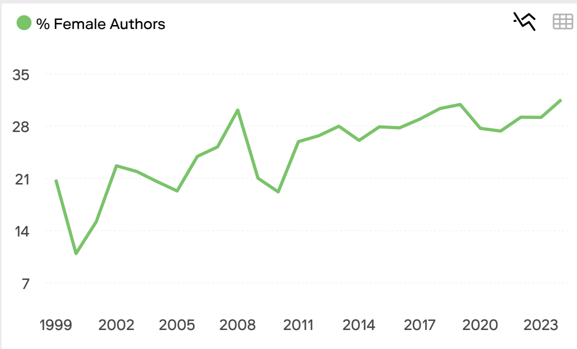

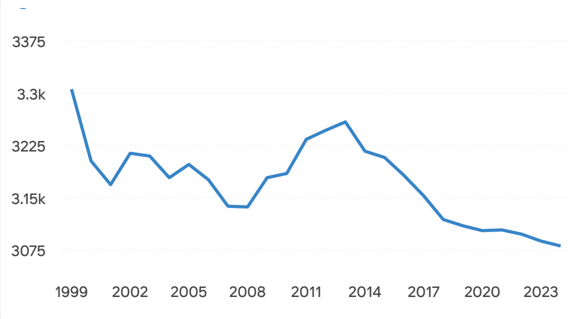

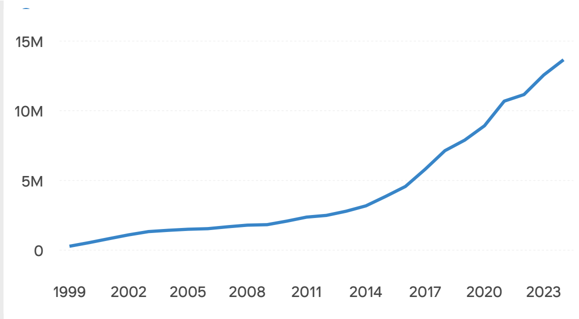

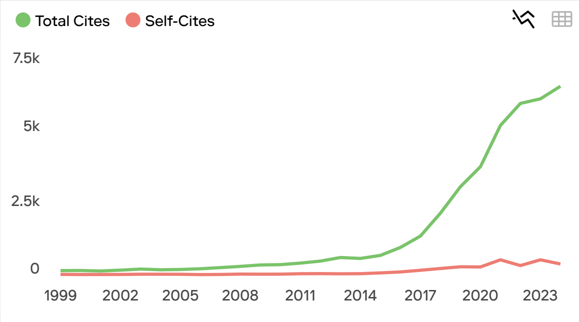

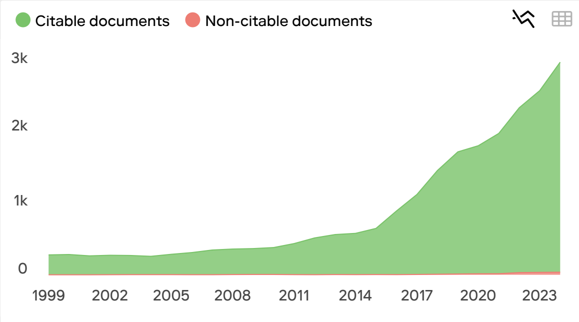

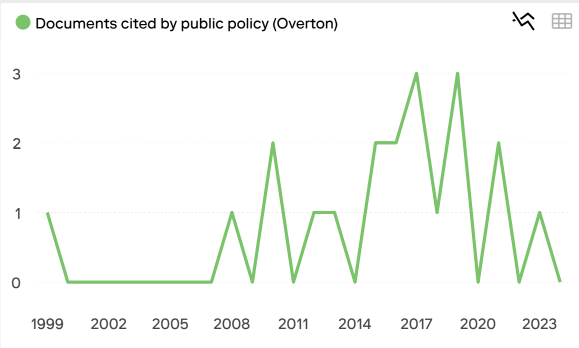

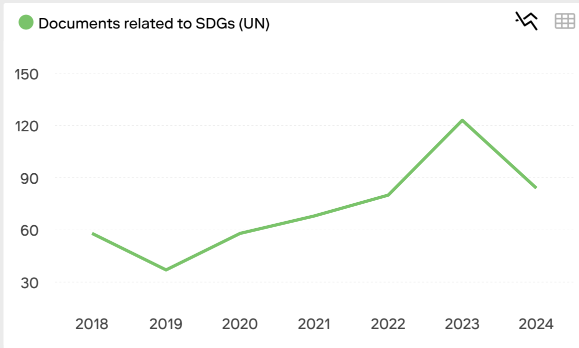

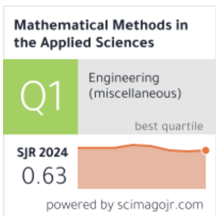

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Golden Ratio-Inspired Subgradient Extragradient Algorithms With Increasing Self-Adaptive Step Sizes for Solving Equilibrium Problems and Applications to Image Restoration

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ABSTRACT

Equilibrium problems (EPs) provide a unified mathematical framework encompassing a broad class of models in optimization, variational inequalities, game theory, and applied sciences. In this paper, we propose two novel subgradient extragradient algorithms inspired by the *golden ratio technique* (GRT) for solving EPs in real Hilbert spaces. Both algorithms employ computationally efficient projections onto suitably constructed half-spaces rather than full projections onto the feasible set, thereby reducing the per-iteration computational cost. A key feature of our schemes is a self-adaptive step-size rule with increasing behavior, which updates the step sizes dynamically without requiring any prior knowledge of Lipschitz-type constants. The first algorithm integrates golden-ratio-based extrapolation with subgradient projection steps, while the second incorporates an alternating extrapolation mechanism to further enhance numerical stability and efficiency. Under standard assumptions, we establish weak convergence of the generated sequences to a solution of the EP, and we additionally prove R-linear convergence under stronger conditions. Extensive numerical experiments, including applications to image restoration, confirm that the proposed methods consistently outperform several existing extragradient-type algorithms in terms of convergence speed, accuracy, and stability.

MSC2020 Classification: 65J15, 47H05, 47J20, 47J25, 90C25

1 | Introduction

The development of efficient and robust iterative methods for solving equilibrium problems remains a central theme in nonlinear analysis and optimization. The study of *equilibrium problems* (EPs) has received significant attention owing

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to their ability to unify and generalize a broad range of mathematical models, including those arising in optimization, variational inequalities, complementarity problems, and game theory.

Formally, let $\mathcal{H}$ be a real Hilbert space equipped with the inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$. Let $C \subseteq \mathcal{H}$ be a nonempty, closed, and convex set. Given a bifunction $\mathcal{F} : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ satisfying $\mathcal{F}(x, x) = 0$ for all $x \in C$, the equilibrium problem is to determine a point $x^* \in C$ such that

$$\mathcal{F}(x^*, x) \geq 0, \quad \forall\, x \in C. \tag{EP}$$

The formulation (EP) encompasses a wide class of models. For instance, when $\mathcal{F}(x, y) = \langle \mathcal{A}(x),\ y - x \rangle$ for a monotone operator $\mathcal{A}$, one recovers the classical *variational inequality problem* (VIP). Likewise, when $\mathcal{F}(x, y) = f(y) - f(x)$ for a convex function f, problem (EP) reduces to a standard convex optimization problem. Many models in economics and applied sciences can also be reformulated as equilibrium problems, including Nash equilibrium models in noncooperative game theory, traffic and transportation equilibrium models, resource allocation, and market equilibrium problems. Classical examples include Cournot's model of oligopolistic competition [1], the Arrow–Debreu general equilibrium theory [2], and Nash's characterization of strategic equilibria [3]. The extensive applicability of EPs has motivated substantial research from both theoretical and computational perspectives; see, for example, [4–9].

Given their importance, various iterative methods have been proposed to solve (EP). These approaches can generally be classified into two main categories:

i. *direct methods*, which reformulate the problem as a fixed-point equation, and

ii. *indirect methods*, which employ projection- or penalty-based iterative approximation schemes.

Among these, the *extragradient-type algorithms* have emerged as particularly effective. Originally introduced by Korpelevich [10] for variational inequalities, the extragradient method was later extended to EPs by Flåm and Antipin [11] and by Tran et al. [12]. This method employs two projection steps per iteration, which considerably enhance convergence properties compared with classical projection algorithms.

More precisely, given an initial point $x_0 \in C$ and a stepsize $\lambda > 0$, the classical extragradient method generates sequences $\{x_k\}$ and $\{y_k\}$ according to

$$\begin{cases} y_k = \arg\min\limits_{x \in C} \left\{ \lambda\, \mathcal{F}(x_k, x) + \frac{1}{2}\|x - x_k\|^2 \right\}, \\ x_{k+1} = \arg\min\limits_{x \in C} \left\{ \lambda\, \mathcal{F}(y_k, x) + \frac{1}{2}\|x - x_k\|^2 \right\}. \end{cases} \tag{1}$$

The convergence of (1) typically requires pseudomonotonicity of $\mathcal{F}$ together with a Lipschitz-type condition [13]. Specifically, there exist constants $c_1, c_2 > 0$ such that

$$\mathcal{F}(x_1, x_3) \leq \mathcal{F}(x_1, x_2) + \mathcal{F}(x_2, x_3) + c_1\|x_1 - x_2\|^2 + c_2\|x_2 - x_3\|^2, \quad \forall\, x_1, x_2, x_3 \in C. \tag{2}$$

Despite its broad applicability, the extragradient method faces two practical limitations:

i. it requires prior knowledge of the Lipschitz-type constants c_1 and c_2 to select an appropriate stepsize λ, which is often impractical; and

ii. in infinite-dimensional Hilbert spaces, the method guarantees only weak convergence and typically exhibits sublinear convergence rates.

To address these limitations, substantial research has explored acceleration strategies. Among the most influential is the *inertial technique*, first introduced by Polyak [14] through the heavy-ball method and later refined by Nesterov [15]. Inertial methods enhance convergence by incorporating momentum-like terms based on previous iterates. For a sequence $\{x_k\}$, a typical inertial step takes the form

$$x_k + \theta_k(x_k - x_{k-1}),$$

where θ_k denotes the inertial parameter.

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Inertial strategies have been successfully integrated into numerous algorithms in optimization and fixed-point theory. Within the context of EPs, inertial terms can substantially reduce both the iteration count and computational cost (see, for example, [16–19]). Further developments are presented in [17, 19–29], with additional contributions in [30–37].

A complementary and increasingly popular line of research is the *golden ratio technique* (GRT), introduced by Malitsky [38]. The GRT provides an elegant extrapolation framework based on the golden ratio constant, allowing for larger step sizes while preserving convergence. Following this development, several golden-ratio-based algorithms have been proposed for EPs and related optimization models [39, 40]. These works demonstrate that carefully designed extrapolation schemes can significantly improve numerical performance without compromising theoretical guarantees.

Another critical factor influencing algorithmic efficiency is the choice of stepsize. Classical approaches often employ decreasing or pre-specified stepsize rules that ensure convergence but may hinder practical performance. Recent advances highlight the advantages of *increasing self-adaptive stepsize strategies*, in which the stepsize is updated dynamically using local information rather than conservative global estimates. Such strategies eliminate the need for prior knowledge of Lipschitz constants and often lead to faster and more stable convergence across a wide range of applications.

Motivated by these developments, we propose two new algorithms combining the strengths of the subgradient extragradient method, the golden ratio technique, and an increasing self-adaptive stepsize rule. The first scheme, Algorithm 1, employs golden-ratio extrapolation together with two resolvent-type steps, one performed on the feasible set C and the other on a suitably constructed half-space. The second scheme, Algorithm 2, introduces an alternating extrapolation strategy in which the golden-ratio update is applied at odd iterations, whereas even iterations rely directly on the current iterate. In both methods, the stepsize is updated adaptively based on local information, allowing controlled increases during the iterative process.

The main contributions of this paper are as follows:

- We introduce a *self-adaptive two-subgradient extragradient method* (Algorithm 1) that integrates golden-ratio extrapolation with two subgradient-based projection steps, achieving strong theoretical guarantees together with practical computational efficiency.
- We propose an *alternating self-adaptive two-subgradient extragradient method* (Algorithm 2), which alternates between golden-ratio extrapolation and direct updates, preserving Fejér monotonicity and enhancing numerical stability.
- Both algorithms incorporate a novel increasing self-adaptive stepsize rule that eliminates the need for prior knowledge of Lipschitz constants, thereby extending applicability to large-scale and ill-conditioned problems.
- We establish weak convergence and R-linear convergence of the proposed algorithms under standard assumptions and validate their efficiency through extensive numerical experiments, including applications to image restoration. The results clearly demonstrate the superiority of the proposed methods in accuracy, stability, and computational efficiency.

The remainder of the paper is organized as follows. Section 2 introduces essential preliminaries and technical tools. Section 3 presents the proposed algorithms and their convergence analysis. Finally, Section 4 provides numerical experiments illustrating the effectiveness of the methods and their application to image restoration.

2 | Preliminaries

This section reviews several fundamental concepts and results that will be used throughout the paper. We denote by $x_k \rightharpoonup x$ the *weak convergence* of a sequence $\{x_k\}$ to x, and by $x_k \to x$ the *strong convergence* of $\{x_k\}$ to x.

The *normal cone* to a convex set $C \subseteq \mathcal{H}$ at a point $x \in C$ is denoted by $N_C(x)$. For a convex function $f : C \to \mathbb{R}$, the *subdifferential* of f at $x \in C$ is written as $\partial f(x)$. The *metric projection* of $x \in \mathcal{H}$ onto C is defined by สำเนาถูกต้อง

$$P_C(x) := \arg\min_{y \in C} \|x - y\|.$$

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The set of all weak cluster points of a sequence $\{x_k\}$ is denoted by

$$w_\omega(\{x_k\}) := \left\{ x \in \mathcal{H} \;\middle|\; \exists\, \{x_{k_j}\} \subset \{x_k\} \text{ such that } x_{k_j} \rightharpoonup x \right\}.$$

Lemma 2.1 ([41, 42]). *Let $C \subset \mathcal{H}$ be a nonempty, closed, and convex set, and let P_C denote the metric projection. Then, for all $x, y \in \mathcal{H}$ and $w \in C$, the following properties hold:*

i. $\|P_C(x) - P_C(y)\|^2 \leq \langle P_C(x) - P_C(y),\ x - y\rangle$;
ii. $z = P_C(x)$ *if and only if* $z \in C$ *and* $\langle x - z,\ z - w\rangle \geq 0$ *for all* $w \in C$;
iii. $\|P_C(x) - w\|^2 + \|x - P_C(x)\|^2 \leq \|x - w\|^2$;
iv. *Let* $Q := \{z \in \mathcal{H} | \langle y,\ z - x\rangle \leq 0\}$ *be a closed half-space with* $y \neq 0$. *Then, for any* $u \in \mathcal{H}$,

$$P_Q(u) = u - \max\left\{ 0,\ \frac{\langle y,\ u - x\rangle}{\|y\|^2} \right\} y.$$

The following basic identities and inequalities hold for any $x, y \in \mathcal{H}$ and $c \in \mathbb{R}$:

1. $\|x + y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2$;
2. $\|x + y\|^2 \leq \|x\|^2 + 2\langle y,\ x + y\rangle$;
3. $\|cx + (1 - c)y\|^2 = c\|x\|^2 + (1 - c)\|y\|^2 - c(1 - c)\|x - y\|^2$.

Definition 2.2 ([5, 43]). Let $\mathcal{F} : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ be a bifunction defined on a nonempty subset $C \subseteq \mathcal{H}$. Let $\varsigma > 0$ and $\eta > 0$. Then $\mathcal{F}$ is said to be:

a. strongly monotone if

$$\mathcal{F}(x, y) + \mathcal{F}(y, x) \leq -\varsigma\|x - y\|^2, \quad \forall\ x, y \in C;$$

b. *monotone* if

$$\mathcal{F}(x, y) + \mathcal{F}(y, x) \leq 0, \quad \forall\ x, y \in C;$$

c. strongly pseudomonotone if

$$\mathcal{F}(x, y) \geq 0 \implies \mathcal{F}(y, x) \leq -\eta\|x - y\|^2, \quad \forall\ x, y \in C;$$

d. *pseudomonotone* if

$$\mathcal{F}(x, y) \geq 0 \implies \mathcal{F}(y, x) \leq 0, \quad \forall\ x, y \in C.$$

The relationships among these notions can be summarized as follows:

$$\text{(a)} \implies \text{(b)} \implies \text{(d)}, \qquad \text{(a)} \implies \text{(c)} \implies \text{(d)}.$$

Lemma 2.3 ([44]). *Let $f : C \to \mathbb{R}$ be a lower semicontinuous convex function. A point $x \in C$ is a minimizer of f over C if and only if*

$$0 \in \partial f(x) + N_C(x),$$

where $\partial f(x)$ denotes the subdifferential of f at x, and $N_C(x)$ is the normal cone to C at x.

Lemma 2.4 ([45]). *Let $\{a_k\}$ and $\{b_k\}$ be sequences of nonnegative real numbers. If, for all $k \in \mathbb{N}$,*

$$a_{k+1} \leq a_k + b_k \quad \text{and} \quad \sum_{k=1}^{+\infty} b_k < +\infty,$$

then limit $\lim_{k\to+\infty} a_k$ exists.

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Lemma 2.5 ([39]). *Let $\{a_k\}_{k\in\mathbb{N}}$ and $\{b_k\}_{k\in\mathbb{N}}$ be sequences of nonnegative real numbers satisfying*

$$a_{k+1} \le a_k - b_k, \quad \forall\, k \ge N,$$

for some nonnegative integer N. Then the following statements hold:

i. $\lim_{k\to+\infty} b_k = 0$;
ii. $\lim_{k\to+\infty} a_k$ *exists and is finite*.

Lemma 2.6 ([46]). *Let $\mathcal{H}$ be a real Hilbert space, and let $\{x_k\} \subset \mathcal{H}$ be a sequence. Suppose there exists a nonempty, closed set $C \subset \mathcal{H}$ such that*:

i. *For every $z \in C$, the limit $\lim_{k\to+\infty} \|x_k - z\|$ exists*;
ii. *Every weak cluster point of $\{x_k\}$ belongs to C.*

Then there exists $\bar{z} \in C$ such that $x_k \rightharpoonup \bar{z}$.

3 | Main Results

In this section, we present new algorithms for solving the equilibrium problem defined in (EP). Starting from two initial points $x_0, x_1 \in \mathcal{H}$, the algorithms generate a sequence $\{x_k\}$ according to the iterative procedures described in Algorithms 1 and 2.

Assumption 3.1. Throughout the analysis, the following conditions are assumed to hold:

F1. $\mathcal{F}(x, x) = 0$ for all $x \in C$.

F2. The solution set $EP(C, \mathcal{F})$ is nonempty.

F3. The bifunction $\mathcal{F}$ is pseudomonotone on C.

F4. The bifunction $\mathcal{F}$ satisfies the Lipschitz-type condition

$$\mathcal{F}(x_1, x_3) \le \mathcal{F}(x_1, x_2) + \mathcal{F}(x_2, x_3) + c_1\|x_1 - x_2\|^2 + c_2\|x_2 - x_3\|^2, \quad \forall\, x_1, x_2, x_3 \in \mathcal{H},$$

for some constants $c_1, c_2 > 0$.

F5. For each fixed $x \in \mathcal{H}$, the mapping

$$y \mapsto \mathcal{F}(x, y)$$

is convex, lower semicontinuous, and subdifferentiable on $\mathcal{H}$.

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F6. For each fixed $y \in \mathcal{H}$, the mapping

$$x \mapsto \mathcal{F}(x, y)$$

is sequentially weakly upper semicontinuous on $\mathcal{H}$; that is, if $\{x_k\} \subset C$ converges weakly to $x \in \mathcal{H}$, then

$$\limsup_{k\to\infty} \mathcal{F}(x_k, y) \le \mathcal{F}(x, y).$$

We now present our first iterative scheme, which constitutes a central contribution of this work. The method integrates the subgradient extragradient framework with golden-ratio-inspired extrapolation and a self-adaptive step-size strategy. The combination of these components yields a robust and convergent algorithm. The detailed procedure is given in Algorithm 1.

Remark 3.2. To enhance the flexibility of step-size adjustment, we refine the conventional self-adaptive update rule by introducing two additional positive parameters, κ_1 and κ_2. When $\kappa_1 = \kappa_2 = 1$, the scheme reduces to the classical two-step subgradient extragradient method. In the general case, choosing $\kappa_1, \kappa_2 > 0$ enables the step size to be adaptively tuned to different problem settings, thereby facilitating faster convergence and improving computational efficiency.

ALGORITHM 1 | Self-adaptive subgradient extragradient method.

1: **Initialization:**
- Choose initial points $w_0, x_1 \in \mathcal{H}$.
- Select parameters $\varphi \in (1, +\infty)$, $\kappa_1, \kappa_2 > 0$, $\lambda_1 > 0$, and $\mu > 0$.
- Define the following sequences:
  (i) $\{p_k\} \subset [0, +\infty)$ with $\sum_{k=1}^{\infty} p_k < +\infty$;
  (ii) $\{\delta_k\} \subset [1, +\infty)$ with $\sum_{k=1}^{\infty} (\delta_k - 1) < +\infty$;
  (iii) $\{\alpha_k\} \subset (0, 1)$.

2: **for** $k = 1, 2, \ldots$ **do**

3: *Step 1 (golden-ratio extrapolation and resolvent on C):*

$$\begin{cases} w_k = \dfrac{\varphi - 1}{\varphi}\, x_k + \dfrac{1}{\varphi}\, w_{k-1}, \\ y_k \in \underset{y \in C}{\arg\min} \left\{ \kappa_1 \lambda_k\, \mathcal{F}(w_k, y) + \frac{1}{2} \|y - w_k\|^2 \right\}. \end{cases}$$

4: *Step 2 (half-space construction):* Pick $\omega_k \in \partial_2 \mathcal{F}(w_k, y_k)$ such that $w_k - \kappa_1 \lambda_k\, \omega_k - y_k \in N_C(y_k)$, and define $\mathcal{H}_k := \left\{ z \in \mathcal{H} \;\middle|\; \langle w_k - \kappa_1 \lambda_k\, \omega_k - y_k,\, z - y_k \rangle \le 0 \right\}$.

5: *Step 3 (second projection step):* $z_k = \arg\min_{y \in \mathcal{H}_k} \left\{ \kappa_2 \lambda_k\, \mathcal{F}(y_k, y) + \frac{1}{2} \|w_k - y\|^2 \right\}$.

6: *Step 4 (convex combination update):* $x_{k+1} = (1 - \alpha_k)\, w_k + \alpha_k\, z_k$.

7: *Step 5 (adaptive step-size update):*

$$\lambda_{k+1} = \begin{cases} \max \left\{ \delta_k \lambda_k + p_k,\ \dfrac{2\mu \left(\mathcal{F}(w_k, z_k) - \mathcal{F}(w_k, y_k) - \mathcal{F}(y_k, z_k) \right)}{\|w_k - y_k\|^2 + \|z_k - y_k\|^2} \right\}, & \text{if } \|w_k - y_k\|^2 + \|z_k - y_k\|^2 > 0, \\ \delta_k \lambda_k + p_k, & \text{otherwise.} \end{cases} \tag{3}$$

8: **end for**

Before proceeding to the main convergence analysis, we first examine the stepsize sequence $\{\lambda_k\}$ generated by the adaptive update rule (3). Establishing that $\{\lambda_k\}$ is well-defined, bounded, monotone, and convergent is crucial for guaranteeing both the well-posedness and the practical implementability of the proposed algorithms. The following lemma summarizes these properties and serves as a foundational step for the subsequent analysis.

Lemma 3.3. *Suppose that the bifunction $\mathcal{F} : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ is of Lipschitz-type with constants $c_1, c_2 > 0$. Let the sequence of stepsizes $\{\lambda_k\}$ be generated by the update rule (3) in Algorithm 1 (or equivalently by (51) in Algorithm 2). Then the following statements hold:*

a. *The sequence $\{\lambda_k\}$ is well-defined for all $k \ge 1$.*

b. *Set*

$$\bar{\delta} := \prod_{j=1}^{\infty} \delta_j \in [1, +\infty), \qquad P := \sum_{k=1}^{\infty} p_k < \infty, \qquad C := 2\mu \max\{c_1, c_2\},$$

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and define

$$\Lambda_1 := \bar{\delta}\,(\lambda_1 + P), \qquad \Lambda_2 := \bar{\delta}\,(C + P), \qquad \Lambda := \max\{\Lambda_1, \Lambda_2\}.$$

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Then $0 < \lambda_k \le \Lambda$ for all $k \ge 1$.

c. *The sequence $\{\lambda_k\}$ converges to a finite limit $\lambda_* \in [\lambda_1, \Lambda]$.*

Proof. (a) Fix $k \ge 1$. If $\|w_k - y_k\|^2 + \|z_k - y_k\|^2 > 0$, then the denominator in the update rule is strictly positive. By the Lipschitz-type property, the numerator

$$\mathcal{F}(w_k, z_k) - \mathcal{F}(w_k, y_k) - \mathcal{F}(y_k, z_k)$$

is finite, and hence the quotient is well-defined and finite. Since $\delta_k \lambda_k + p_k$ is also finite, their maximum is finite, which ensures that λ_{k+1} is well-defined. If $\|w_k - y_k\|^2 + \|z_k - y_k\|^2 = 0$, the update reduces to $\lambda_{k+1} = \delta_k \lambda_k + p_k$, which is again finite. Therefore, $\{\lambda_k\}$ is well-defined for all $k \geq 1$.

(b) From the Lipschitz-type condition we obtain, whenever $\|w_k - y_k\|^2 + \|z_k - y_k\|^2 > 0$,

$$\frac{\mathcal{F}(w_k, z_k) - \mathcal{F}(w_k, y_k) - \mathcal{F}(y_k, z_k)}{\|w_k - y_k\|^2 + \|z_k - y_k\|^2} \leq \max\{c_1, c_2\},$$

hence

$$\frac{2\mu\big(\mathcal{F}(w_k, z_k) - \mathcal{F}(w_k, y_k) - \mathcal{F}(y_k, z_k)\big)}{\|w_k - y_k\|^2 + \|z_k - y_k\|^2} \leq C := 2\mu \max\{c_1, c_2\}.$$

Consequently, in all cases,

$$\lambda_{k+1} = \begin{cases} \max\{\delta_k \lambda_k + p_k,\ Q_k\}, & \text{if the denominator is positive,} \\ \delta_k \lambda_k + p_k, & \text{otherwise,} \end{cases}$$

where $Q_k \leq C$. Since $\delta_k \geq 1$ and $p_k \geq 0$, we have

$$\lambda_{k+1} \geq \delta_k \lambda_k + p_k \geq \lambda_k,$$

so $\{\lambda_k\}$ is nondecreasing and $\lambda_k \geq \lambda_1 > 0$ for all k.

To prove the uniform upper bound, we analyze the possible patterns of the update. There are two cases.

Case 1: For a given $k \geq 1$, the second term in the maximum of (3) is never selected for indices $1, \ldots, k-1$; that is, $\lambda_{j+1} = \delta_j \lambda_j + p_j$ for all $j = 1, \ldots, k-1$. Then, by iterating this recursion, we obtain

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$$\lambda_k = \left(\prod_{j=1}^{k-1} \delta_j\right)\lambda_1 + \sum_{i=1}^{k-1}\left(\prod_{j=i+1}^{k-1} \delta_j\right) p_i \leq \bar{\delta}\ \lambda_1 + \bar{\delta}\ P = \Lambda_1,$$

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since $\prod_{j=i+1}^{k-1} \delta_j \leq \prod_{j=1}^{\infty} \delta_j = \bar{\delta}$ and $\sum_{i=1}^{\infty} p_i = P$.

Case 2: For this k, there exists at least one index $j \in \{1, \ldots, k-1\}$ at which the second term in the maximum is selected. Let m be the largest such index, that is, the last index in $\{1, \ldots, k-1\}$ for which

$$\lambda_{m+1} = \max\left\{\delta_m \lambda_m + p_m,\ \frac{2\mu(\mathcal{F}(w_m, z_m) - \mathcal{F}(w_m, y_m) - \mathcal{F}(y_m, z_m))}{\|w_m - y_m\|^2 + \|z_m - y_m\|^2}\right\}$$

is attained by the second term. Then $\lambda_{m+1} \leq C$. Moreover, by the choice of m, for all $j = m+1, \ldots, k-1$ the update uses only the first term, that is, $\lambda_{j+1} = \delta_j \lambda_j + p_j$. Starting from index $m+1$, we can therefore write, for $k \geq m+1$,

$$\lambda_k = \left(\prod_{j=m+1}^{k-1} \delta_j\right)\lambda_{m+1} + \sum_{i=m+1}^{k-1}\left(\prod_{j=i+1}^{k-1} \delta_j\right) p_i \leq \bar{\delta}\ \lambda_{m+1} + \bar{\delta}\ P \leq \bar{\delta}\ (C + P) = \Lambda_2.$$

Combining both cases, we conclude that $\lambda_k \leq \max\{\Lambda_1, \Lambda_2\} = \Lambda$ for all $k \geq 1$. Together with $\lambda_k \geq \lambda_1 > 0$, this proves (b).

(c) Since $\{\lambda_k\}$ is nondecreasing and bounded above by Λ, it converges by the Monotone Convergence Theorem. Its limit λ_* lies in $[\lambda_1, \Lambda]$, which completes the proof. □

Before establishing the convergence of the proposed algorithms, we first derive a fundamental inequality that characterizes the descent behavior of the intermediate sequence $\{z_k\}$. This inequality plays a central role in the convergence analysis, as it provides a recursive estimate that links the iterates, the bifunction $\mathcal{F}$, and the adaptive step sizes.

Lemma 3.4. *Let $\{x_k\}$ be the sequence generated by Algorithm 1 with the adaptive step-size rule (3), and suppose that Assumption 3.1 holds. Then, for any solution $x^* \in EP(C, \mathcal{F})$ and all $k \geq 1$, the following inequality holds:*

$$\|z_k - x^*\|^2 \leq \|w_k - x^*\|^2 - \left(1 - \frac{\kappa_2}{\kappa_1}\right)\|w_k - z_k\|^2$$
$$- \frac{\kappa_2}{\kappa_1}\left(1 - \frac{\kappa_1}{\mu}\ \lambda_k \lambda_{k+1}\right)\|w_k - y_k\|^2 - \frac{\kappa_2}{\kappa_1}\left(1 - \frac{\kappa_1}{\mu}\ \lambda_k \lambda_{k+1}\right)\|z_k - y_k\|^2. \tag{4}$$

Proof. Here, $\partial_2 \mathcal{F}(u, \cdot)$ denotes the subdifferential of the convex function $v \mapsto \mathcal{F}(u, v)$ (Assumption 3.1F5), and $N_C(\cdot)$ denotes the normal cone to a closed convex set C.

By Lemma 2.3, applied to the minimization in Step 3, there exist $v \in \partial_2 \mathcal{F}(y_k, z_k)$ and $\bar{v} \in N_{\mathcal{H}_k}(z_k)$ such that

$$\kappa_2 \lambda_k\ v + z_k - w_k + \bar{v} = 0. \tag{5}$$

Taking the inner product of (5) with $y - z_k$ and using $\langle \bar{v}, y - z_k \rangle \leq 0$ for all $y \in \mathcal{H}_k$ yields

$$\langle w_k - z_k,\ y - z_k \rangle \leq \kappa_2 \lambda_k\ \langle v,\ y - z_k \rangle, \quad \forall\ y \in \mathcal{H}_k. \tag{6}$$

By the subgradient inequality for $\mathcal{F}(y_k, \cdot)$,

$$\mathcal{F}(y_k, y) - \mathcal{F}(y_k, z_k) \geq \langle v,\ y - z_k \rangle, \qquad \forall\ y \in \mathcal{H}. \tag{7}$$

Combining expressions (6) and (7) gives

$$\kappa_2 \lambda_k \big[\mathcal{F}(y_k, y) - \mathcal{F}(y_k, z_k)\big] \geq \langle w_k - z_k,\ y - z_k \rangle, \qquad \forall\ y \in \mathcal{H}_k. \tag{8}$$

By Step 2 of the algorithm, there exists $\omega_k \in \partial_2 \mathcal{F}(w_k, y_k)$ such that $w_k - \kappa_1 \lambda_k \omega_k - y_k \in N_C(y_k)$. Since $z_k \in \mathcal{H}_k$, we obtain

$$\langle w_k - y_k,\ z_k - y_k \rangle \leq \kappa_1 \lambda_k\ \langle \omega_k,\ z_k - y_k \rangle. \tag{9}$$

By the subgradient inequality for $\mathcal{F}(w_k, \cdot)$,

$$\mathcal{F}(w_k, z_k) - \mathcal{F}(w_k, y_k) \geq \langle \omega_k,\ z_k - y_k \rangle, \tag{10}$$

and hence

$$\kappa_1 \lambda_k \big[\mathcal{F}(w_k, z_k) - \mathcal{F}(w_k, y_k)\big] \geq \langle w_k - y_k,\ z_k - y_k \rangle. \tag{11}$$

Taking $y = x^*$ in (8) and applying Assumption 3.1 F3 with $x^* \in EP(C, \mathcal{F})$ gives $\mathcal{F}(y_k, x^*) \leq 0$, hence

$$\langle w_k - z_k,\ z_k - x^* \rangle \geq \kappa_2 \lambda_k\ \mathcal{F}(y_k, z_k). \tag{12}$$

From (3), whenever $\|w_k - y_k\|^2 + \|z_k - y_k\|^2 > 0$, we have

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$$\lambda_{k+1} \geq \frac{2\mu\big(\mathcal{F}(w_k, z_k) - \mathcal{F}(w_k, y_k) - \mathcal{F}(y_k, z_k)\big)}{\|w_k - y_k\|^2 + \|z_k - y_k\|^2},$$

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or equivalently,

$$\mathcal{F}(w_k, z_k) - \mathcal{F}(w_k, y_k) - \mathcal{F}(y_k, z_k) \leq \frac{\lambda_{k+1}}{2\mu}\big(\|w_k - y_k\|^2 + \|z_k - y_k\|^2\big). \tag{13}$$

Multiplying (13) by $\kappa_2 \lambda_k$ and rearranging yields

$$\kappa_2 \lambda_k\ \mathcal{F}(y_k, z_k) \geq \kappa_2 \lambda_k\ \mathcal{F}(w_k, z_k) - \kappa_2 \lambda_k\ \mathcal{F}(w_k, y_k) - \frac{\kappa_2 \lambda_k \lambda_{k+1}}{2\mu}\big(\|w_k - y_k\|^2 + \|z_k - y_k\|^2\big). \tag{14}$$

Combining (12–14) with (11) gives

$$\langle w_k - z_k,\ z_k - x^* \rangle \geq \frac{\kappa_2}{\kappa_1} \langle w_k - y_k,\ z_k - y_k \rangle - \frac{\kappa_2 \lambda_k \lambda_{k+1}}{2\mu} \left(\|w_k - y_k\|^2 + \|z_k - y_k\|^2 \right).$$

Using the Hilbert space identities

$$2\langle w_k - z_k,\ z_k - x^* \rangle = \|w_k - x^*\|^2 - \|w_k - z_k\|^2 - \|z_k - x^*\|^2,$$

and

$$2\langle w_k - y_k,\ z_k - y_k \rangle = \|w_k - y_k\|^2 + \|z_k - y_k\|^2 - \|w_k - z_k\|^2,$$

multiplying the previous inequality by 2 and rearranging yields (4).

If $\|w_k - y_k\|^2 + \|z_k - y_k\|^2 = 0$, then $w_k = y_k = z_k$ and (4) holds trivially. □

Remark 3.5. For Fejér-type convergence arguments, it is convenient that the coefficients of the squared norms in (4) are nonnegative. This is ensured by assuming $\kappa_2 \leq \kappa_1$ and, additionally,

$$\frac{\kappa_1}{\mu} \lambda_k \lambda_{k+1} \leq 1.$$

Theorem 3.6 (Weak convergence of Algorithm 1). *Let $\mathcal{H}$ be a real Hilbert space, and let $C \subset \mathcal{H}$ be a nonempty, closed, and convex subset. Suppose that the bifunction $\mathcal{F} : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ satisfies Assumption 3.1* (*i.e., conditions* (*F1–F5*)), *and that the control parameters fulfill the following requirements*:

a. *The sequence of stepsizes $\{\lambda_k\}$ generated by Algorithm 1 satisfies*

$$0 < \lambda_1 \leq \lambda_k \leq \Lambda < \infty, \qquad \lim_{k \to \infty} \lambda_k = \lambda_*,$$

for some constant $\Lambda > 0$.

b. *The following* ***control conditions*** *are imposed*:

$$\kappa_2 \leq \kappa_1, \qquad \text{(ensures a nonnegative coefficient in front of } \|w_k - z_k\|^2\text{)}, \tag{C1}$$

$$\mu > \kappa_1 \Lambda^2. \tag{C2}$$

c. *There exist constants* $0 < \underline{\alpha} \leq \overline{\alpha} < 1$ *such that*

$$\underline{\alpha} \leq \alpha_k \leq \overline{\alpha}, \qquad \forall k \in \mathbb{N}.$$

Then the sequence $\{x_k\}$ generated by Algorithm 1 converges weakly to some $x^* \in EP(C, \mathcal{F})$.

Proof. We divide the proof into two parts.

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(Part A) Fejér-type inequality and vanishing residuals.

By Lemma 3.4, for every $x^* \in EP(C, \mathcal{F})$, we have

$$\begin{aligned} \|z_k - x^*\|^2 \leq \|w_k - x^*\|^2 &- \left(1 - \frac{\kappa_2}{\kappa_1}\right) \|w_k - z_k\|^2 \\ &- \frac{\kappa_2}{\kappa_1} \left(1 - \frac{\kappa_1}{\mu}\ \lambda_k \lambda_{k+1}\right) \left(\|w_k - y_k\|^2 + \|z_k - y_k\|^2 \right). \end{aligned} \tag{15}$$

From the convex combination update in Algorithm 1,

$$x_{k+1} = (1 - \alpha_k)\ w_k + \alpha_k\ z_k,$$

the standard identity $\|(1-\alpha)a + \alpha b\|^2 = (1-\alpha)\|a\|^2 + \alpha\|b\|^2 - \alpha(1-\alpha)\|a-b\|^2$ gives

$$\begin{aligned} \|x_{k+1} - x^*\|^2 &= \left\|(1-\alpha_k)(w_k - x^*) + \alpha_k(z_k - x^*)\right\|^2 \\ &= (1-\alpha_k)\|w_k - x^*\|^2 + \alpha_k\|z_k - x^*\|^2 - \alpha_k(1-\alpha_k)\|w_k - z_k\|^2. \end{aligned} \tag{16}$$

Substituting (15) into (16) yields

$$\begin{aligned} \|x_{k+1} - x^*\|^2 \le \|w_k - x^*\|^2 &- \left[\alpha_k\left(1 - \frac{\kappa_2}{\kappa_1}\right) + \alpha_k(1-\alpha_k)\right]\|w_k - z_k\|^2 \\ &- \alpha_k \frac{\kappa_2}{\kappa_1}\left(1 - \frac{\kappa_1}{\mu}\ \lambda_k\lambda_{k+1}\right)\left(\|w_k - y_k\|^2 + \|z_k - y_k\|^2\right). \end{aligned} \tag{17}$$

Define

$$\begin{aligned} D_k \ := &\left[\alpha_k\left(1 - \frac{\kappa_2}{\kappa_1}\right) + \alpha_k(1-\alpha_k)\right]\|w_k - z_k\|^2 \\ &+ \alpha_k \frac{\kappa_2}{\kappa_1}\left(1 - \frac{\kappa_1}{\mu}\ \lambda_k\lambda_{k+1}\right)\left(\|w_k - y_k\|^2 + \|z_k - y_k\|^2\right). \end{aligned} \tag{18}$$

By (C1), $\kappa_2 \le \kappa_1$ and $0 < \alpha_k < 1$, hence $\alpha_k\left(1 - \frac{\kappa_2}{\kappa_1}\right) + \alpha_k(1-\alpha_k) = \alpha_k\left(2 - \alpha_k - \frac{\kappa_2}{\kappa_1}\right) > 0$. Moreover, by (C2) and (a), we have $\lambda_k\lambda_{k+1} \le \Lambda^2 < \frac{\mu}{\kappa_1} \ \Rightarrow \ 1 - \frac{\kappa_1}{\mu}\lambda_k\lambda_{k+1} > 0$, so the second coefficient in (18) is also nonnegative. Therefore, $D_k \ge 0, \ \forall k \in \mathbb{N}$.

Consequently, for all $k \ge 1$,

$$\|x_{k+1} - x^*\|^2 \le \|w_k - x^*\|^2 - D_k. \tag{19}$$

From the golden-ratio extrapolation step,

$$w_k = \frac{\varphi - 1}{\varphi} x_k + \frac{1}{\varphi}\ w_{k-1}, \qquad (\varphi > 1),$$

we obtain the identity

$$\|w_k - x^*\|^2 = \frac{\varphi - 1}{\varphi}\ \|x_k - x^*\|^2 + \frac{1}{\varphi}\ \|w_{k-1} - x^*\|^2 - \frac{\varphi - 1}{\varphi^2}\ \|x_k - w_{k-1}\|^2. \tag{20}$$

Set $\phi \ := \varphi$, and introduce the Lyapunov functional

$$\Psi_k \ := \|x_k - x^*\|^2 + \frac{1}{\phi - 1}\ \|w_{k-1} - x^*\|^2, \qquad k \ge 1. \tag{21}$$

Combining (19) and (20), we obtain

$$\Psi_{k+1} \le \Psi_k - \frac{1}{\phi}\ \|x_k - w_{k-1}\|^2 - D_k. \tag{22}$$

Since $\Psi_k \ge 0$ and $D_k \ge 0$ for all k, the sequence $\{\Psi_k\}$ is nonincreasing and bounded below, and therefore convergent. Summing (22) over k yields

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$$\sum_{k=1}^{\infty}\left(\frac{1}{\phi}\ \|x_k - w_{k-1}\|^2 + D_k\right) < \infty.$$

Hence,

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$$\|x_k - w_{k-1}\| \ \to \ 0, \qquad D_k \ \to \ 0 \quad \text{as } k \to \infty. \tag{23}$$

From the definition of D_k in (18) and the fact that both coefficients are strictly positive, it follows that

$$\|w_k - z_k\| \ \to \ 0, \qquad \|w_k - y_k\| \ \to \ 0, \qquad \|z_k - y_k\| \ \to \ 0, \tag{24}$$

as $k \to \infty$.

Thus, Part A establishes a Fejér-type inequality together with the asymptotic vanishing of the key residual terms.

(Part B) Opial's conditions and identification of weak cluster points.

(O1) Existence of the limit of the distance to every solution. Fix an arbitrary $x^* \in EP(C, \mathcal{F})$. From (22), $\{\Psi_k\}$ is nonincreasing and bounded below, hence there exists $\Psi_* \geq 0$ such that $\Psi_k \to \Psi_*$ as $k \to \infty$. Moreover,

$$\sum_{k=1}^{\infty} \left(\frac{1}{\phi} \, \|x_k - w_{k-1}\|^2 + D_k \right) < \infty$$

implies again

$$\|x_k - w_{k-1}\| \;\to\; 0, \qquad D_k \;\to\; 0. \tag{25}$$

Using the golden-ratio relations

$$w_k - w_{k-1} = \frac{\phi - 1}{\phi}(x_k - w_{k-1}), \qquad x_k - w_k = \frac{1}{\phi - 1}(w_k - w_{k-1}),$$

we further deduce

$$\|w_k - w_{k-1}\| \;\to\; 0, \qquad \|x_k - w_k\| \;\to\; 0. \tag{26}$$

Consequently,

$$\Big| \|w_k - x^*\|^2 - \|x_k - x^*\|^2 \Big| \leq \big(\|w_k - x^*\| + \|x_k - x^*\| \big) \, \|w_k - x_k\| \;\to\; 0. \tag{27}$$

Since

$$\Psi_k = \|x_k - x^*\|^2 + \frac{1}{\phi - 1} \|w_{k-1} - x^*\|^2$$

converges and

$$\|w_{k-1} - x^*\|^2 - \|x_{k-1} - x^*\|^2 \;\to\; 0$$

by (27), it follows that the sequence $\{\|x_k - x^*\|\}$ has a limit for every $x^* \in EP(C, \mathcal{F})$. Moreover, since

$$\|x_k - x^*\|^2 \leq \Psi_k \leq \Psi_1,$$

the sequence $\{x_k\}$ is bounded. This establishes Opial's condition (O1) and the boundedness required for Opial's lemma.

(O2) Every weak cluster point is a solution. Let $\hat{x}$ be a weak cluster point of $\{x_k\}$, and consider a subsequence $\{x_{k_j}\}$ with $x_{k_j} \rightharpoonup \hat{x}$. From (24) and (26), we obtain

$$\|x_{k_j} - w_{k_j}\| \to 0, \qquad \|w_{k_j} - y_{k_j}\| \to 0, \qquad \|w_{k_j} - z_{k_j}\| \to 0,$$

which implies $w_{k_j} \rightharpoonup \hat{x}$ and, consequently,

$$y_{k_j} \rightharpoonup \hat{x}, \qquad z_{k_j} \rightharpoonup \hat{x}. \tag{28}$$

Since $y_{k_j}, z_{k_j} \in C$ and C is weakly closed, it follows that $\hat{x} \in C$. From the half-space optimality condition (see (8)) and the inclusion $C \subseteq \mathcal{H}_k$, we obtain, for all $y \in C$,

$$\kappa_2 \lambda_k \big[\mathcal{F}(y_k, y) - \mathcal{F}(y_k, z_k) \big] \geq \langle w_k - z_k, \; y - z_k \rangle. \tag{29}$$

Using Lemma 3.4 (in particular, inequalities analogous to (12–14)) and the definition of D_k, we can bound $\mathcal{F}(y_k, z_k)$ from below to obtain

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$$\begin{aligned} \kappa_2 \lambda_k \; \mathcal{F}(y_k, y) &\geq \kappa_2 \lambda_k \; \mathcal{F}(w_k, z_k) - \kappa_2 \lambda_k \; \mathcal{F}(w_k, y_k) - \frac{\kappa_2 \lambda_k \lambda_{k+1}}{2\mu} \big(\|w_k - y_k\|^2 + \|z_k - y_k\|^2 \big) \\ &\quad + \langle w_k - z_k, \; y - z_k \rangle \\ &\geq \frac{\kappa_2}{\kappa_1} \, \langle w_k - y_k, \; z_k - y_k \rangle - \frac{\kappa_2 \lambda_k \lambda_{k+1}}{2\mu} \big(\|w_k - y_k\|^2 + \|z_k - y_k\|^2 \big) + \langle w_k - z_k, \; y - z_k \rangle. \end{aligned} \tag{30}$$

Passing to the subsequence $k = k_j$ and using the vanishing residuals

$$\|w_{k_j} - y_{k_j}\| \to 0, \qquad \|z_{k_j} - y_{k_j}\| \to 0, \qquad \|w_{k_j} - z_{k_j}\| \to 0,$$

together with the boundedness of $\{x_{k_j}\}$, the right-hand side of (30) tends to 0. Since $\lambda_{k_j} \geq \lambda_1 > 0$, we conclude that

$$\liminf_{j\to\infty} \mathcal{F}(y_{k_j}, y) \geq 0, \qquad \forall\ y \in C. \tag{31}$$

By Assumption 3.1(F6) (sequential weak upper semicontinuity of the mapping $x \mapsto \mathcal{F}(x, y)$ for each fixed y), together with $y_{k_j} \rightharpoonup \hat{x}$ from (28), we obtain

$$\limsup_{j\to\infty} \mathcal{F}(y_{k_j}, y) \leq \mathcal{F}(\hat{x}, y), \qquad \forall\ y \in C. \tag{32}$$

Combining (31) and (32), we deduce that $\mathcal{F}(\hat{x}, y) \geq 0$ for all $y \in C$, and hence $\hat{x} \in EP(C, \mathcal{F})$. This establishes Opial's condition (O2).

We have thus verified both conditions (O1) and (O2) of Opial's lemma, and shown that $\{x_k\}$ is bounded. Therefore, by Opial's lemma, the sequence $\{x_k\}$ converges weakly to some $x^* \in EP(C, \mathcal{F})$, which completes the proof. □

Next, we establish the R-linear convergence of the sequence generated by Algorithm 1 under suitable conditions.

Lemma 3.7 (R-linear convergence). *Let $\{x_k\}$ be the sequence generated by Algorithm 1, and suppose that Assumption 3.1 holds. Assume further that*:

a. *The stepsizes $\{\lambda_k\}$ satisfy*
$$0 < \lambda_1 \leq \lambda_k \leq \Lambda < +\infty, \qquad \lambda_k \to \lambda_* > 0,$$
for some constant $\Lambda > 0$.

b. *The control parameters satisfy*
$$\kappa_2 \leq \kappa_1 \quad \text{and} \quad \mu > \kappa_1 \Lambda^2.$$

c. *There exist constants $0 < \underline{\alpha} \leq \overline{\alpha} < 1$ such that*
$$\underline{\alpha} \leq \alpha_k \leq \overline{\alpha}, \qquad \forall k \in \mathbb{N}.$$

d. *The bifunction $\mathcal{F}$ is η-strongly pseudomonotone on C with parameter $\eta > 0$ (see Definition 2.2c).*

Then the equilibrium problem $EP(\mathcal{F}, C)$ has a unique solution $x^* \in C$, and the sequence $\{x_k\}$ converges R-linearly to x^*.

Proof. Let $x_1, x_2 \in EP(\mathcal{F}, C)$. Then $\mathcal{F}(x_1, x_2) \geq 0$ and $\mathcal{F}(x_2, x_1) \geq 0$. Since $\mathcal{F}$ is η-strongly pseudomonotone, from $\mathcal{F}(x_1, x_2) \geq 0$ we obtain

$$\mathcal{F}(x_2, x_1) \leq -\eta \|x_2 - x_1\|^2.$$

Combining with $\mathcal{F}(x_2, x_1) \geq 0$ yields $\|x_2 - x_1\| = 0$, hence $x_1 = x_2$. Thus $EP(\mathcal{F}, C) = \{x^*\}$ is a singleton.

By strong pseudomonotonicity (Definition 2.2c), we have

$$\mathcal{F}(x^*, y_k) \geq 0 \implies \mathcal{F}(y_k, x^*) \leq -\,\eta\ \|y_k - x^*\|^2, \qquad \forall\ y_k \in C. \tag{33}$$

Since $x^* \in EP(\mathcal{F}, C)$, we have $\mathcal{F}(x^*, y_k) \geq 0$ for all $y_k \in C$. Thus

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$$\mathcal{F}(y_k, x^*) \leq -\,\eta\ \|y_k - x^*\|^2. \tag{34}$$

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From (8) (half-space optimality condition in Lemma 3.4), we have

$$\kappa_2 \lambda_k \big[\mathcal{F}(y_k, y) - \mathcal{F}(y_k, z_k)\big] \geq \langle w_k - z_k,\ y - z_k\rangle, \qquad \forall\ y \in \mathcal{H}_k. \tag{35}$$

Since $C \subseteq \mathcal{H}_k$ (supporting half-space construction) and $x^* \in C$, we can take $y = x^*$ in (35) and obtain

$$\kappa_2 \lambda_k \big[\mathcal{F}(y_k, x^*) - \mathcal{F}(y_k, z_k)\big] \geq \langle w_k - z_k,\ x^* - z_k \rangle.$$

Equivalently,

$$\langle w_k - z_k,\ z_k - x^* \rangle \geq \kappa_2 \lambda_k \big[\mathcal{F}(y_k, z_k) - \mathcal{F}(y_k, x^*)\big].$$

Using (34), we get

$$\langle w_k - z_k,\ z_k - x^* \rangle \geq \kappa_2 \lambda_k \big(\mathcal{F}(y_k, z_k) + \eta\, \|y_k - x^*\|^2\big). \tag{36}$$

Repeating the derivation in Lemma 3.4 but now keeping the extra term $\eta\|y_k - x^*\|^2$ from (36), and using the stepsize bound from (13), one obtains (instead of (4)) the refined inequality

$$\begin{aligned} \|x_{k+1} - x^*\|^2 \leq \|w_k - x^*\|^2 - \alpha_k \left(1 - \frac{\kappa_2}{\kappa_1}\right) \|w_k - z_k\|^2 \\ - \alpha_k \frac{\kappa_2}{\kappa_1} \left(1 - \frac{\kappa_1}{\mu}\ \lambda_k \lambda_{k+1}\right) \big(\|w_k - y_k\|^2 + \|z_k - y_k\|^2\big) - 2\kappa_2 \eta \lambda_k \|y_k - x^*\|^2. \end{aligned} \tag{37}$$

Using $\mu > \kappa_1 \Lambda^2$ and $\lambda_k \lambda_{k+1} \leq \Lambda^2$, we have

$$\frac{\kappa_1}{\mu}\ \lambda_k \lambda_{k+1} \leq \frac{\kappa_1 \Lambda^2}{\mu} < 1, \tag{38}$$

and hence

$$1 - \frac{\kappa_1}{\mu}\ \lambda_k \lambda_{k+1} > 0. \tag{39}$$

Together with $\kappa_2 \leq \kappa_1$ and $0 < \underline{\alpha} \leq \alpha_k \leq \overline{\alpha} < 1$, this ensures that all coefficients in front of the squared norms in (37) are nonnegative. Discarding the nonpositive term involving $\|w_k - z_k\|^2$ (to simplify), we obtain

$$\begin{aligned} \|x_{k+1} - x^*\|^2 \leq \|w_k - x^*\|^2 - \alpha_k \frac{\kappa_2}{\kappa_1} \left(1 - \frac{\kappa_1}{\mu}\ \lambda_k \lambda_{k+1}\right) \|w_k - y_k\|^2 \\ - 2\kappa_2 \eta \lambda_k \|y_k - x^*\|^2. \end{aligned} \tag{40}$$

Define

$$A_k \ \triangleq\ \alpha_k \frac{\kappa_2}{\kappa_1} \left(1 - \frac{\kappa_1}{\mu} \lambda_k \lambda_{k+1}\right), \qquad B_k \ \triangleq\ 2\kappa_2 \eta \lambda_k,$$

so that

$$\|x_{k+1} - x^*\|^2 \leq \|w_k - x^*\|^2 - A_k\ \|w_k - y_k\|^2 - B_k\ \|y_k - x^*\|^2. \tag{41}$$

Since $\lambda_k \to \lambda_* > 0$ and $\mu > \kappa_1 \Lambda^2$, we have

$$\Upsilon_k \ \triangleq\ 1 - \frac{\kappa_1}{\mu} \lambda_k \lambda_{k+1} \to \Upsilon_\infty \ \triangleq\ 1 - \frac{\kappa_1}{\mu} \lambda_*^2 > 0.$$

Thus there exists $k_0 \in \mathbb{N}$ and positive constants

$$\Upsilon \in (0, \Upsilon_\infty], \qquad \underline{\lambda} \in (0, \lambda_*],$$

such that, for all $k \geq k_0$,

$$\Upsilon_k \geq \Upsilon, \qquad \lambda_k \geq \underline{\lambda}.$$

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Using $\alpha_k \geq \underline{\alpha}$, we obtain

$$A_k \geq \underline{\alpha}\ \frac{\kappa_2}{\kappa_1}\ \Upsilon \ \triangleq\ A_0 > 0, \qquad B_k \geq 2\kappa_2 \eta\ \underline{\lambda} \ \triangleq\ B_0 > 0, \quad \forall k \geq k_0. \tag{42}$$

Using the basic Hilbert space inequality $\|u\|^2 + \|v\|^2 \geq \frac{1}{2}\ \|u + v\|^2,\ \forall\, u, v \in \mathcal{H}$, with $u = w_k - y_k$ and $v = y_k - x^*$, we get

$$\|w_k - y_k\|^2 + \|y_k - x^*\|^2 \geq \frac{1}{2}\ \|w_k - x^*\|^2. \tag{43}$$

Combining (41), (42) and (43), for all $k \geq k_0$ we obtain

$$
\begin{aligned}
\|x_{k+1} - x^*\|^2 &\leq \|w_k - x^*\|^2 - \min\{A_k, B_k\}\left(\|w_k - y_k\|^2 + \|y_k - x^*\|^2\right) \\
&\leq \left(1 - \frac{1}{2}\ \min\{A_0, B_0\}\right) \|w_k - x^*\|^2.
\end{aligned} \tag{44}
$$

Define the contraction factor

$$
\rho^2 \ \triangleq\ 1 - \frac{1}{2}\min\{A_0, B_0\} \in (0,1),
$$

so that

$$
\|x_{k+1} - x^*\|^2 \leq \rho^2\ \|w_k - x^*\|^2, \qquad \forall\ k \geq k_0. \tag{45}
$$

Now use the golden-ratio extrapolation relation for w_{k+1}:

$$
w_{k+1} = \frac{\varphi - 1}{\varphi} x_{k+1} + \frac{1}{\varphi} w_k, \qquad \varphi > 1.
$$

Applying the same identity as in (20) and dropping the negative term, we obtain

$$
\frac{\varphi}{\varphi - 1}\ \|w_{k+1} - x^*\|^2 \leq \|x_{k+1} - x^*\|^2 + \frac{1}{\varphi - 1}\ \|w_k - x^*\|^2. \tag{46}
$$

Combining (45) and (46), for all $k \geq k_0$ we have

$$
\begin{aligned}
\frac{\varphi}{\varphi - 1}\ \|w_{k+1} - x^*\|^2 &\leq \rho^2\ \|w_k - x^*\|^2 + \frac{1}{\varphi - 1}\ \|w_k - x^*\|^2 \\
&= \left(\rho^2 + \frac{1}{\varphi - 1}\right) \|w_k - x^*\|^2.
\end{aligned} \tag{47}
$$

Hence,

$$
\|w_{k+1} - x^*\|^2 \leq \delta^2\ \|w_k - x^*\|^2, \qquad \forall\ k \geq k_0, \tag{48}
$$

where

$$
\delta^2 \ \triangleq\ \frac{\rho^2 + \frac{1}{\varphi - 1}}{\varphi\varphi - 1} = \frac{(\varphi - 1)\rho^2 + 1}{\varphi}, \qquad \delta \in (0,1).
$$

By induction, for all $k \geq k_0$,

$$
\|w_{k+1} - x^*\|^2 \leq \delta^{\ 2[(k+1) - k_0]}\ \|w_{k_0} - x^*\|^2. \tag{49}
$$

Finally, combining (45) with (49), we obtain

$$
\|x_{k+1} - x^*\|^2 \leq \rho^2\ \delta^{\ 2(k - k_0)}\ \|w_{k_0} - x^*\|^2, \qquad \forall\ k \geq k_0. \tag{50}
$$

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Taking the k-th root on both sides of (50) and passing to the limit gives

$$
\limsup_{k \to \infty} \|x_{k+1} - x^*\|^{1/k} \leq \delta < 1,
$$

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which shows that $\{x_k\}$ converges R-linearly to x^*. □

In addition to Algorithm 1, we propose an alternative scheme that simplifies the computational structure while preserving convergence guarantees. The central idea is to alternate between a golden-ratio extrapolation step and a direct resolvent step. The detailed procedure is presented in Algorithm 2.

Theorem 3.8 (Weak convergence of Algorithm 2). *Let $\mathcal{H}$ be a real Hilbert space, and let $C \subset \mathcal{H}$ be a nonempty, closed, and convex set. Assume that the bifunction $\mathcal{F} : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ satisfies Assumption 3.1, and that the control parameters fulfill the following requirements*:

ALGORITHM 2 | Alternating self-adaptive subgradient extragradient method.

Initialization:

- Choose initial points $w_0, x_1 \in \mathcal{H}$.
- Select parameters $\varphi \in (1, +\infty)$, $\kappa_1, \kappa_2 > 0$, $\lambda_1 > 0$, and $\mu > 0$.
- Define the following sequences:
  (i) $\{p_k\} \subset [0, +\infty)$ with $\sum_{k=1}^{\infty} p_k < +\infty$;
  (ii) $\{\delta_k\} \subset [1, +\infty)$ with $\sum_{k=1}^{\infty} (\delta_k - 1) < +\infty$;
  (iii) $\{\alpha_k\} \subset (0, 1)$.

for $k = 1, 2, \dots$ **do**

Step 1 (alternating golden-ratio extrapolation and resolvent on C):

$$w_k = \begin{cases} \dfrac{\varphi - 1}{\varphi}\, x_k + \dfrac{1}{\varphi}\, w_{k-1}, & \text{if } k \text{ is odd}, \\ x_k, & \text{if } k \text{ is even}, \end{cases} \qquad y_k \in \arg\min_{y \in C} \left\{ \kappa_1 \lambda_k\, \mathcal{F}(w_k, y) + \tfrac{1}{2} \|y - w_k\|^2 \right\}.$$

if $w_k = y_k$ **then**

stop (under (F0)–(F5), $w_k \in \mathrm{EP}(C, \mathcal{F})$).

end if

Step 2 (half-space construction): Pick $\omega_k \in \partial_2 \mathcal{F}(w_k, y_k)$ such that $w_k - \kappa_1 \lambda_k\, \omega_k - y_k \in N_C(y_k)$, and define $\mathcal{H}_k := \left\{ z \in \mathcal{H} \mid \langle w_k - \kappa_1 \lambda_k\, \omega_k - y_k,\ z - y_k \rangle \le 0 \right\}$.

Step 3 (second projection step): $z_k = \arg\min_{y \in \mathcal{H}_k} \left\{ \kappa_2 \lambda_k\, \mathcal{F}(y_k, y) + \tfrac{1}{2} \|w_k - y\|^2 \right\}$.

Step 4 (convex combination update): $x_{k+1} = (1 - \alpha_k)\, w_k + \alpha_k\, z_k$.

Step 5 (adaptive step-size update):

$$\lambda_{k+1} = \begin{cases} \max \left\{ \delta_k \lambda_k + p_k,\ \dfrac{2\mu \left(\mathcal{F}(w_k, z_k) - \mathcal{F}(w_k, y_k) - \mathcal{F}(y_k, z_k) \right)}{\|w_k - y_k\|^2 + \|z_k - y_k\|^2} \right\}, & \text{if } \|w_k - y_k\|^2 + \|z_k - y_k\|^2 > 0, \\ \delta_k \lambda_k + p_k, & \text{otherwise}. \end{cases} \tag{51}$$

end for

a. *The sequence $\{\lambda_k\}$ generated by Algorithm 2 satisfies*

$$0 < \lambda_1 \le \lambda_k \le \Lambda < \infty, \qquad \lim_{k \to \infty} \lambda_k = \lambda_*,$$

for some constant $\Lambda > 0$.

b. *The following **control conditions** are imposed:*

$$\kappa_2 \le \kappa_1, \qquad (\textit{ensures nonnegativity of the coefficient of } \|w_k - z_k\|^2), \tag{C1$'$}$$

$$\textit{either } \mu > \kappa_1 \Lambda^2 \quad \text{or} \quad \limsup_{k \to \infty} \lambda_k < \sqrt{\frac{\mu}{\kappa_1}}. \tag{C2$'$}$$

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c. *There exist constants $0 < \underline{\alpha} \le \overline{\alpha} < 1$ such that*

$$\underline{\alpha} \le \alpha_k \le \overline{\alpha}, \qquad \forall\, k \in \mathbb{N}.$$

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Then the sequence $\{x_k\}$ generated by Algorithm 2 converges weakly to some $x^* \in EP(\mathcal{F}, C)$.

Proof. The proof follows the same pattern as Theorem 3.6, with an additional even/odd index decomposition.

Step 1: One-step Fejér-type inequality.

As in Lemma 3.4, for every $x^* \in EP(\mathcal{F}, C)$ and all $k \geq 1$, we have

$$\|z_k - x^*\|^2 \leq \|w_k - x^*\|^2 - \left(1 - \frac{\kappa_2}{\kappa_1}\right)\|w_k - z_k\|^2 - \frac{\kappa_2}{\kappa_1}\left(1 - \frac{\kappa_1}{\mu}\,\lambda_k\lambda_{k+1}\right)\left(\|w_k - y_k\|^2 + \|z_k - y_k\|^2\right). \tag{52}$$

Moreover, from the convex combination update,

$$x_{k+1} = (1 - \alpha_k)\, w_k + \alpha_k\, z_k,$$

we obtain

$$\|x_{k+1} - x^*\|^2 = (1 - \alpha_k)\, \|w_k - x^*\|^2 + \alpha_k\, \|z_k - x^*\|^2 - \alpha_k(1 - \alpha_k)\, \|w_k - z_k\|^2. \tag{53}$$

Substituting (52) into (53) yields, for all $k \geq 1$,

$$\|x_{k+1} - x^*\|^2 \leq \|w_k - x^*\|^2 - \left[\alpha_k\left(1 - \frac{\kappa_2}{\kappa_1}\right) + \alpha_k(1 - \alpha_k)\right]\|w_k - z_k\|^2 - \alpha_k\frac{\kappa_2}{\kappa_1}\left(1 - \frac{\kappa_1}{\mu}\,\lambda_k\lambda_{k+1}\right)\left(\|w_k - y_k\|^2 + \|z_k - y_k\|^2\right). \tag{54}$$

Step 2: Even subsequence $\{x_{2m}\}$ and basic descent.

For even indices, Algorithm 2 gives $w_{2m} = x_{2m}$. Specializing (54) to $k = 2m$ and using $w_{2m} = x_{2m}$, we obtain

$$\|x_{2m+1} - x^*\|^2 - \|x_{2m} - x^*\|^2 \leq -\left[\alpha_{2m}\left(1 - \frac{\kappa_2}{\kappa_1}\right) + \alpha_{2m}(1 - \alpha_{2m})\right]\|x_{2m} - z_{2m}\|^2 - \alpha_{2m}\frac{\kappa_2}{\kappa_1}\left(1 - \frac{\kappa_1}{\mu}\,\lambda_{2m}\lambda_{2m+1}\right)\left(\|x_{2m} - y_{2m}\|^2 + \|z_{2m} - y_{2m}\|^2\right). \tag{55}$$

Define the nonnegative decrement

$$S_{2m} := \left[\alpha_{2m}\left(1 - \frac{\kappa_2}{\kappa_1}\right) + \alpha_{2m}(1 - \alpha_{2m})\right]\|x_{2m} - z_{2m}\|^2 + \alpha_{2m}\frac{\kappa_2}{\kappa_1}\left(1 - \frac{\kappa_1}{\mu}\,\lambda_{2m}\lambda_{2m+1}\right)\left(\|x_{2m} - y_{2m}\|^2 + \|z_{2m} - y_{2m}\|^2\right) \geq 0. \tag{56}$$

Then (55) can be written as

$$\|x_{2m+1} - x^*\|^2 - \|x_{2m} - x^*\|^2 \leq -\,S_{2m} \leq 0, \qquad m \geq 1. \tag{57}$$

Step 3: Odd indices and a two-step descent for $\{x_{2m}\}$.

For odd indices $k = 2m + 1$, Algorithm 2 gives

$$w_{2m+1} = \frac{\varphi - 1}{\varphi}x_{2m+1} + \frac{1}{\varphi}\, w_{2m} = \frac{\varphi - 1}{\varphi}x_{2m+1} + \frac{1}{\varphi}\, x_{2m},$$

and therefore

$$\|w_{2m+1} - x^*\|^2 = \frac{\varphi - 1}{\varphi}\, \|x_{2m+1} - x^*\|^2 + \frac{1}{\varphi}\, \|x_{2m} - x^*\|^2 - \frac{\varphi - 1}{\varphi^2}\, \|x_{2m+1} - x_{2m}\|^2. \tag{58}$$

Substituting (58) into (54) with $k = 2m + 1$ yields

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$$\|x_{2m+2} - x^*\|^2 \leq \frac{\varphi - 1}{\varphi}\, \|x_{2m+1} - x^*\|^2 + \frac{1}{\varphi}\, \|x_{2m} - x^*\|^2 - \frac{\varphi - 1}{\varphi^2}\, \|x_{2m+1} - x_{2m}\|^2 - \left[\alpha_{2m+1}\left(1 - \frac{\kappa_2}{\kappa_1}\right) + \alpha_{2m+1}(1 - \alpha_{2m+1})\right]\|w_{2m+1} - z_{2m+1}\|^2 - \alpha_{2m+1}\frac{\kappa_2}{\kappa_1}\left(1 - \frac{\kappa_1}{\mu}\,\lambda_{2m+1}\lambda_{2m+2}\right)\left(\|w_{2m+1} - y_{2m+1}\|^2 + \|z_{2m+1} - y_{2m+1}\|^2\right). \tag{59}$$

Subtracting $\|x_{2m} - x^*\|^2$ from both sides and using (57) (which implies $\|x_{2m+1} - x^*\|^2 - \|x_{2m} - x^*\|^2 \le -S_{2m}$), we deduce

$$
\begin{aligned}
\|x_{2m+2} - x^*\|^2 - \|x_{2m} - x^*\|^2 \le &- \frac{\varphi - 1}{\varphi}\, S_{2m} - \frac{\varphi - 1}{\varphi^2}\, \|x_{2m+1} - x_{2m}\|^2 \\
&- \left[\alpha_{2m+1}\left(1 - \frac{\kappa_2}{\kappa_1}\right) + \alpha_{2m+1}(1 - \alpha_{2m+1})\right] \|w_{2m+1} - z_{2m+1}\|^2 \\
&- \alpha_{2m+1}\frac{\kappa_2}{\kappa_1}\left(1 - \frac{\kappa_1}{\mu}\, \lambda_{2m+1}\lambda_{2m+2}\right)\left(\|w_{2m+1} - y_{2m+1}\|^2 + \|z_{2m+1} - y_{2m+1}\|^2\right) \\
\le &\; 0,
\end{aligned} \tag{60}
$$

for all $m \ge 1$.

Step 4: *Monotonicity of the even subsequence and vanishing residuals.*

Inequality (60) shows that the *even subsequence* $\{\|x_{2m} - x^*\|^2\}_{m\ge1}$ is nonincreasing and bounded below, hence convergent. In particular,

$$
\lim_{m\to\infty} \|x_{2m} - x^*\| \;\text{ exists, and therefore }\; \{x_{2m}\} \text{ is bounded.} \tag{61}
$$

Summing (60) over m from 1 to N and letting $N \to \infty$, we obtain

$$
\begin{aligned}
\sum_{m=1}^{\infty} \Bigg(&\frac{\varphi - 1}{\varphi}\, S_{2m} + \frac{\varphi - 1}{\varphi^2}\, \|x_{2m+1} - x_{2m}\|^2 + \left[\alpha_{2m+1}\left(1 - \frac{\kappa_2}{\kappa_1}\right) + \alpha_{2m+1}(1 - \alpha_{2m+1})\right] \|w_{2m+1} - z_{2m+1}\|^2 \\
&+ \alpha_{2m+1}\frac{\kappa_2}{\kappa_1}\left(1 - \frac{\kappa_1}{\mu}\, \lambda_{2m+1}\lambda_{2m+2}\right)\left(\|w_{2m+1} - y_{2m+1}\|^2 + \|z_{2m+1} - y_{2m+1}\|^2\right)\Bigg) < \infty.
\end{aligned}
$$

By condition (C2$'$) and boundedness of $\{\lambda_k\}$, there exist $\varepsilon > 0$ and $m_0 \in \mathbb{N}$ such that

$$
1 - \frac{\kappa_1}{\mu}\, \lambda_k \lambda_{k+1} \ge \varepsilon, \qquad \forall\; k \ge 2m_0.
$$

Together with $\alpha_k \in [\underline{\alpha}, \overline{\alpha}] \subset (0, 1)$, this implies that all coefficients in the infinite sum above are eventually bounded below by positive constants. Hence,

$$
\|x_{2m+1} - x_{2m}\| \to 0, \quad \|w_{2m+1} - z_{2m+1}\| \to 0, \quad \|w_{2m+1} - y_{2m+1}\| \to 0, \quad \|z_{2m+1} - y_{2m+1}\| \to 0. \tag{62}
$$

Moreover, since $S_{2m} \to 0$ as $m \to \infty$, its definition (56), together with the same positivity argument for the coefficients, gives

$$
\|x_{2m} - z_{2m}\| \to 0, \qquad \|x_{2m} - y_{2m}\| \to 0, \qquad \|z_{2m} - y_{2m}\| \to 0. \tag{63}
$$

Step 5: Limit of the distance to any solution (Opial's condition (O1)).

From (62), using

$$
w_{2m+1} = \frac{\varphi - 1}{\varphi} x_{2m+1} + \frac{1}{\varphi}\, x_{2m},
$$

we have

$$
w_{2m+1} - x_{2m+1} = \frac{1}{\varphi}(x_{2m} - x_{2m+1}),
$$

and hence

$$
\|w_{2m+1} - x_{2m+1}\| = \frac{1}{\varphi}\, \|x_{2m+1} - x_{2m}\| \;\to\; 0. \tag{64}
$$

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Combining (62), (63), and (64), we conclude that

$$
\|x_k - w_k\| \to 0, \quad \|w_k - y_k\| \to 0, \quad \|w_k - z_k\| \to 0,
$$

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along both even and odd indices.

As a consequence of (61) and (62),

$$\Big| \|x_{2m+1} - x^*\| - \|x_{2m} - x^*\| \Big| \le \|x_{2m+1} - x_{2m}\| \;\to\; 0,$$

so the odd subsequence $\{\|x_{2m+1} - x^*\|\}$ converges to the same limit as the even subsequence $\{\|x_{2m} - x^*\|\}$. Therefore the full sequence $\{\|x_k - x^*\|\}_{k\ge1}$ admits a finite limit for every $x^* \in EP(\mathcal{F}, C)$. This verifies Opial's condition (O1).

Step 6: *Identification of weak cluster points (Opial's condition (O2)).*

Let $\hat{x}$ be a weak cluster point of $\{x_k\}$ and consider a subsequence $x_{k_j} \rightharpoonup \hat{x}$. From the residual vanishing just established,

$$\|x_{k_j} - w_{k_j}\| \to 0, \quad \|w_{k_j} - y_{k_j}\| \to 0, \quad \|w_{k_j} - z_{k_j}\| \to 0,$$

we deduce

$$w_{k_j} \rightharpoonup \hat{x}, \qquad y_{k_j} \rightharpoonup \hat{x}, \qquad z_{k_j} \rightharpoonup \hat{x}.$$

Since $y_{k_j}, z_{k_j} \in C$ and C is weakly closed, it follows that $\hat{x} \in C$.

From the half-space inequality (8) and the inclusion $C \subseteq \mathcal{H}_k$, we have, for all $y \in C$ and all $k \ge 1$,

$$\kappa_2\lambda_k\big[\mathcal{F}(y_k, y) - \mathcal{F}(y_k, z_k)\big] \ge \langle w_k - z_k,\; y - z_k\rangle. \tag{65}$$

Using (11) and (14) to bound $\mathcal{F}(y_k, z_k)$ from below, we obtain

$$\begin{aligned}\kappa_2\lambda_k\, \mathcal{F}(y_k, y) \ge{}& \frac{\kappa_2}{\kappa_1}\,\langle w_k - y_k,\; z_k - y_k\rangle - \frac{\kappa_2\lambda_k\lambda_{k+1}}{2\mu}\big(\|w_k - y_k\|^2 + \|z_k - y_k\|^2\big)\\ &+ \langle w_k - z_k,\; y - z_k\rangle.\end{aligned} \tag{66}$$

Passing to the subsequence $k = k_j$ and using the fact that all residual terms vanish while $\{\lambda_k\}$ is bounded and $\lambda_{k_j} \ge \lambda_1 > 0$, the right−hand side of (66) tends to 0, and hence

$$\liminf_{j\to\infty} \mathcal{F}(y_{k_j}, y) \ge 0, \qquad \forall\; y \in C. \tag{67}$$

By Assumption 3.1(F6) (weak upper semicontinuity of the mapping $x \mapsto \mathcal{F}(x, y)$ for each fixed y) and the fact that $y_{k_j} \rightharpoonup \hat{x}$, we also have

$$\limsup_{j\to\infty} \mathcal{F}(y_{k_j}, y) \le \mathcal{F}(\hat{x}, y), \qquad \forall\; y \in C. \tag{68}$$

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Combining (67) and (68) yields

$$\mathcal{F}(\hat{x}, y) \ge 0, \qquad \forall\; y \in C,$$

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that is, $\hat{x} \in EP(\mathcal{F}, C)$. This establishes Opial's condition (O2).

Since both Opial's conditions (O1) and (O2) are satisfied and the sequence $\{x_k\}$ is bounded, Opial's lemma implies that $\{x_k\}$ converges weakly to some $x^* \in EP(\mathcal{F}, C)$, which completes the proof. □

Next, we establish that the sequence generated by Algorithm 2 achieves R-linear convergence under suitable assumptions.

Lemma 3.9 (R-linear convergence of Algorithm 2). *Let $\{x_k\}$ be the sequence generated by Algorithm 2, and suppose that Assumption 3.1 holds. Assume further that the bifunction $\mathcal{F}$ is η-strongly pseudomonotone on C with parameter $\eta > 0$. In addition, suppose that the control parameters satisfy the conditions, that is,*

$$0 < \lambda_1 \le \lambda_k \le \Lambda < \infty, \quad \lambda_k \to \lambda_*, \quad \kappa_2 \le \kappa_1, \quad \mu > \kappa_1\Lambda^2, \quad 0 < \underline{\alpha} \le \alpha_k \le \overline{\alpha} < 1.$$

Then the sequence $\{x_k\}$ converges R-linearly to the unique solution $x^* \in C$ of the equilibrium problem $EP(\mathcal{F}, C)$.

Proof. Since $\mathcal{F}$ is η-strongly pseudomonotone on C and $x^* \in EP(\mathcal{F}, C)$, we have

$$\mathcal{F}(x^*, y) \geq 0 \quad \Longrightarrow \quad \mathcal{F}(y, x^*) \leq -\,\eta\, \|y - x^*\|^2, \qquad \forall\ y \in C.$$

Arguing as in the proof of Lemma 3.4, but now incorporating the strong pseudomonotonicity at x^*, we obtain for all $k \geq 1$,

$$\begin{aligned}
\|x_{k+1} - x^*\|^2 \leq \|w_k - x^*\|^2 &- \left[\alpha_k\left(1 - \frac{\kappa_2}{\kappa_1}\right) + \alpha_k(1-\alpha_k)\right]\|w_k - z_k\|^2 \\
&- \alpha_k\, \frac{\kappa_2}{\kappa_1}\left(1 - \frac{\kappa_1}{\mu}\lambda_k\lambda_{k+1}\right)\left(\|w_k - y_k\|^2 + \|z_k - y_k\|^2\right) \\
&- 2\ \alpha_k\ \kappa_2\ \eta\ \lambda_k\ \|y_k - x^*\|^2.
\end{aligned} \tag{69}$$

For even indices, Algorithm 2 gives $w_{2m} = x_{2m}$. Hence, (69) with $k = 2m$ yields

$$\begin{aligned}
\|x_{2m+1} - x^*\|^2 \leq \|x_{2m} - x^*\|^2 &- \left[\alpha_{2m}\left(1 - \frac{\kappa_2}{\kappa_1}\right) + \alpha_{2m}(1-\alpha_{2m})\right]\|x_{2m} - z_{2m}\|^2 \\
&- \alpha_{2m}\, \frac{\kappa_2}{\kappa_1}\left(1 - \frac{\kappa_1}{\mu}\lambda_{2m}\lambda_{2m+1}\right)\left(\|x_{2m} - y_{2m}\|^2 + \|z_{2m} - y_{2m}\|^2\right) \\
&- 2\ \alpha_{2m}\ \kappa_2\ \eta\ \lambda_{2m}\ \|y_{2m} - x^*\|^2.
\end{aligned} \tag{70}$$

Discarding the nonpositive terms involving $\|x_{2m} - z_{2m}\|^2$ and $\|z_{2m} - y_{2m}\|^2$, we obtain the simpler inequality

$$\|x_{2m+1} - x^*\|^2 \leq \|x_{2m} - x^*\|^2 - A_{2m}\ \|x_{2m} - y_{2m}\|^2 - B_{2m}\ \|y_{2m} - x^*\|^2, \tag{71}$$

where

$$A_{2m} := \alpha_{2m}\, \frac{\kappa_2}{\kappa_1}\left(1 - \frac{\kappa_1}{\mu}\lambda_{2m}\lambda_{2m+1}\right), \qquad B_{2m} := 2\ \alpha_{2m}\ \kappa_2\ \eta\ \lambda_{2m}.$$

From the assumptions of Theorem 3.8 we know that

$$0 < \underline{\alpha} \leq \alpha_k \leq \overline{\alpha} < 1, \quad 0 < \lambda_1 \leq \lambda_k \leq \Lambda < \infty, \quad \mu > \kappa_1\Lambda^2.$$

Thus

$$\frac{\kappa_1}{\mu}\lambda_k\lambda_{k+1} \leq \frac{\kappa_1}{\mu}\Lambda^2 < 1,$$

so there exists $\varepsilon \in (0,1)$ such that

$$1 - \frac{\kappa_1}{\mu}\ \lambda_k\lambda_{k+1} \geq \varepsilon > 0, \qquad \forall\ k \geq 1.$$

In addition, since $\lambda_k \to \lambda_* > 0$, there exists $\underline{\lambda} \in (0, \lambda_*]$ and $k_0 \in \mathbb{N}$ such that $\lambda_k \geq \underline{\lambda}$ for all $k \geq k_0$.

Therefore, for all m with $2m \geq k_0$, we have

$$A_{2m} \geq \underline{\alpha}\ \frac{\kappa_2}{\kappa_1}\varepsilon\ \triangleq\ A_0 > 0, \qquad B_{2m} \geq 2\ \underline{\alpha}\ \kappa_2\ \eta\ \underline{\lambda}\ \triangleq\ B_0 > 0.$$

Using the elementary inequality $\|u\|^2 + \|v\|^2 \geq \frac{1}{2}\ \|u+v\|^2,\ \forall\ u, v \in \mathcal{H}$, with $u = x_{2m} - y_{2m}$ and $v = y_{2m} - x^*$, we obtain

$$\|x_{2m} - y_{2m}\|^2 + \|y_{2m} - x^*\|^2 \geq \frac{1}{2}\|x_{2m} - x^*\|^2.$$

Hence, for all m with $2m \geq k_0$,

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$$A_{2m}\|x_{2m} - y_{2m}\|^2 + B_{2m}\|y_{2m} - x^*\|^2 \geq \frac{1}{2}\min\{A_0, B_0\}\ \|x_{2m} - x^*\|^2.$$

Substituting this into (71) yields

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$$\|x_{2m+1} - x^*\|^2 \leq \rho^2\ \|x_{2m} - x^*\|^2, \qquad \forall\ m \text{ with } 2m \geq k_0, \tag{72}$$

where $\rho^2 := 1 - \frac{1}{2}\min\{A_0, B_0\} \in (0,1)$.

For odd indices, Algorithm 2 gives the golden-ratio relation

$$w_{2m+1} = \frac{\varphi - 1}{\varphi} x_{2m+1} + \frac{1}{\varphi}\, w_{2m} = \frac{\varphi - 1}{\varphi} x_{2m+1} + \frac{1}{\varphi}\, x_{2m}.$$

Using identity (20), we have

$$\|w_{2m+1} - x^*\|^2 = \frac{\varphi - 1}{\varphi}\, \|x_{2m+1} - x^*\|^2 + \frac{1}{\varphi}\, \|x_{2m} - x^*\|^2 - \frac{\varphi - 1}{\varphi^2}\, \|x_{2m+1} - x_{2m}\|^2. \tag{73}$$

Applying (69) with $k = 2m + 1$ and discarding the nonpositive terms on the right-hand side, we obtain

$$\|x_{2m+2} - x^*\|^2 \le \|w_{2m+1} - x^*\|^2.$$

Combining this with (73) and dropping the last term (which is nonpositive), we get

$$\|x_{2m+2} - x^*\|^2 \le \frac{\varphi - 1}{\varphi}\|x_{2m+1} - x^*\|^2 + \frac{1}{\varphi}\, \|x_{2m} - x^*\|^2.$$

Using (72) to bound $\|x_{2m+1} - x^*\|^2$, we obtain

$$\begin{aligned} \|x_{2m+2} - x^*\|^2 &\le \frac{\varphi - 1}{\varphi}\, \rho^2\, \|x_{2m} - x^*\|^2 + \frac{1}{\varphi}\, \|x_{2m} - x^*\|^2 \\ &= \delta^2\, \|x_{2m} - x^*\|^2, \end{aligned}$$

where

$$\delta^2 := \frac{\varphi - 1}{\varphi}\rho^2 + \frac{1}{\varphi} = 1 - \frac{\varphi - 1}{\varphi}(1 - \rho^2) \in (0, 1),$$

since $\rho^2 \in (0, 1)$ and $\varphi > 1$.

Thus, for all m sufficiently large,

$$\|x_{2m+2} - x^*\|^2 \le \delta^2\, \|x_{2m} - x^*\|^2, \qquad 0 < \delta < 1. \tag{74}$$

By iterating (74), we obtain for all $m \ge M$ (for some M large enough),

$$\|x_{2m} - x^*\|^2 \le \delta^{\,2(m-M)}\, \|x_{2M} - x^*\|^2.$$

Using (72) once more, we also have

$$\|x_{2m+1} - x^*\|^2 \le \rho^2\, \|x_{2m} - x^*\|^2 \le \rho^2\, \delta^{\,2(m-M)}\, \|x_{2M} - x^*\|^2.$$

Thus, both subsequences $\{x_{2m}\}$ and $\{x_{2m+1}\}$ converge to x^* at a geometric rate. Equivalently, there exist constants $C > 0$ and $\theta \in (0, 1)$ such that

$$\|x_k - x^*\| \le C\, \theta^k, \qquad \forall\, k \text{ sufficiently large},$$

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which shows that $\{x_k\}$ converges to x^* at an R-linear rate. □

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4 | Numerical Experiments and Performance Evaluation

In this section, we present a set of computational experiments to evaluate and compare the performance of the proposed methods in Section 3 with several benchmark algorithms from the literature. We also analyze structural features of the proposed schemes and investigate how different control parameters influence their numerical behavior.

All implementations were carried out in MATLAB R2022b on a Lenovo laptop equipped with an Intel Core i9-13900H CPU (2.60 GHz) and 32 GB RAM. To ensure a fair comparison, all algorithms were executed under the parameter settings summarized in Table 1. The choices for the proposed methods were selected based on stable empirical performance; they need not satisfy the theoretical sufficient conditions (e.g., $\mu > \kappa_1 \Lambda^2$), since such assumptions are not necessary in practice but only sufficient for convergence guarantees.

TABLE 1 | Control parameters for benchmark methods (ExM1–ExM4) and proposed methods (PM1–PM2).

Algorithm (Abbreviation)	Control parameters
Algorithm 1 in [19] (*ExM1*)	$\theta = 0.65,\ \lambda = \min\left\{\frac{1}{4c_1},\ \frac{1}{4c_2}\right\},\ \epsilon_n = n^{-2}$
Algorithm 2.1 in [18] (*ExM2*)	$\alpha = 0.35,\ \mu = 0.65,\ \tau = \dfrac{0.9(1-\alpha)^2}{\alpha(1+\alpha)+(1-\alpha)^2},\ \lambda_1 = 0.5$
Algorithm 3.3 in [16] (*ExM3*)	$\mu = 0.50,\ \epsilon = 10^{-6},\ \lambda_1 = 0.10,\ \theta_n = \frac{1-\epsilon}{3},\ \phi_n = 0.5$
Algorithm 2.1 in [17] (*ExM4*)	$\lambda_1 = 0.5,\ \mu = 0.60,\ \theta = 0.75 \cdot \frac{1-\mu}{2}$
Algorithm 1 (Proposed Method 1, *PM1*)	$\varphi = 1.95,\ \kappa_1 = 1.0,\ \kappa_2 = 1.0,\ \lambda_1 = 1.2,\ \mu = 0.5,\ p_k = k^{-2},\ \delta_k = 1 + k^{-2},\ \alpha_k = 0.5$
Algorithm 2 (Proposed Method 2, *PM2*)	$\varphi = 1.95,\ \kappa_1 = 1.0,\ \kappa_2 = 1.0,\ \lambda_1 = 1.2,\ \mu = 0.5,\ p_k = k^{-2},\ \delta_k = 1 + k^{-2},\ \alpha_k = 0.5$

Example 4.1. Consider the Nash–Cournot oligopolistic equilibrium model introduced in [12]. The bifunction $\mathcal{F} : \mathbb{R}^5 \times \mathbb{R}^5 \to \mathbb{R}$ is defined by

$$\mathcal{F}(x, y) = \langle Px + Qy + c,\ y - x\rangle,$$

where the matrices P, Q, and the vector c are

$$P = \begin{pmatrix} 3.1 & 2 & 0 & 0 & 0 \\ 2 & 3.6 & 0 & 0 & 0 \\ 0 & 0 & 3.5 & 2 & 0 \\ 0 & 0 & 2 & 3.3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}, \qquad Q = \begin{pmatrix} 1.6 & 1 & 0 & 0 & 0 \\ 1 & 1.6 & 0 & 0 & 0 \\ 0 & 0 & 1.5 & 1 & 0 \\ 0 & 0 & 1 & 1.5 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}, \qquad c = \begin{pmatrix} 1 \\ -2 \\ -1 \\ 2 \\ -1 \end{pmatrix}.$$

It is straightforward to verify that

$$Q - P \quad \text{is symmetric and negative semidefinite,} \qquad Q \quad \text{is symmetric and positive semidefinite.}$$

These properties ensure that $\mathcal{F}$ is pseudomonotone and satisfies the standard Lipschitz-type condition (2). Indeed, the associated Lipschitz-type constants are $c_1 = c_2 = \frac{1}{2}\,\|P - Q\|_2 = 1.4525$, where $\|\cdot\|_2$ denotes the spectral (operator) norm. The feasible set is the box constraint

$$C := \left\{ x \in \mathbb{R}^5 \;\middle|\; -5 \le x_i \le 5,\quad i = 1, \dots, 5 \right\}.$$

We begin our numerical investigation with Example 4.1, which is based on the classical Nash–Cournot oligopolistic equilibrium model. The purpose of this experiment is twofold:

i. to demonstrate the applicability of the proposed algorithms to structured equilibrium problems arising in economics, and

ii. to assess their numerical efficiency relative to existing state-of-the-art methods.

To obtain a fair and representative comparison, we employ six distinct initial points in the Hilbert space $\mathbb{R}^5$:

$$x^{(1)} = (1,\ 1,\ 1,\ 1,\ 1)^\top, \qquad x^{(2)} = (-1,\ -1,\ -1,\ -1,\ -1)^\top,$$
$$x^{(3)} = (2,\ -2,\ 2,\ -2,\ 2)^\top, \qquad x^{(4)} = (3,\ 0,\ -3,\ 0,\ 3)^\top,$$
$$x^{(5)} = (4,\ 2,\ 0,\ -2,\ -4)^\top, \qquad x^{(6)} = (5,\ -4,\ 3,\ -2,\ 1)^\top.$$

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These test points were deliberately selected to include symmetric patterns, alternating-sign structures, sparse vectors, and mixed-sign configurations, ensuring a broad assessment of the numerical behavior across all algorithms.

Numerical observations. From Table 2 and Figures 1 and 2, several important conclusions can be drawn:

TABLE 2 | Numerical performance of all algorithms for Example 4.1 using six initial points $x^{(i)}$, $i = 1, \dots, 6$. Each entry is reported in the format (Iterations, Final Error, Time in seconds).

Method	$x^{(1)}$	$x^{(2)}$	$x^{(3)}$	$x^{(4)}$	$x^{(5)}$	$x^{(6)}$
ExM1	$(47, 7.86 \times 10^{-7}, 0.63)$	$(47, 4.73 \times 10^{-7}, 0.51)$	$(58, 6.95 \times 10^{-7}, 0.66)$	$(60, 5.85 \times 10^{-7}, 0.63)$	$(54, 6.18 \times 10^{-7}, 0.57)$	$(64, 8.30 \times 10^{-7}, 0.69)$
ExM2	$(38, 6.04 \times 10^{-7}, 0.70)$	$(46, 5.77 \times 10^{-7}, 0.76)$	$(43, 6.95 \times 10^{-7}, 0.69)$	$(37, 8.92 \times 10^{-7}, 0.60)$	$(38, 9.40 \times 10^{-7}, 0.60)$	$(43, 6.00 \times 10^{-7}, 0.67)$
ExM3	$(39, 2.58 \times 10^{-7}, 0.64)$	$(36, 2.93 \times 10^{-7}, 0.59)$	$(41, 4.25 \times 10^{-7}, 0.68)$	$(50, 7.03 \times 10^{-7}, 0.81)$	$(63, 2.05 \times 10^{-7}, 0.95)$	$(58, 1.73 \times 10^{-7}, 0.88)$
ExM4	$(31, 7.88 \times 10^{-7}, 0.52)$	$(33, 5.99 \times 10^{-7}, 0.60)$	$(34, 6.01 \times 10^{-7}, 0.56)$	$(32, 9.69 \times 10^{-7}, 0.51)$	$(34, 5.34 \times 10^{-7}, 0.52)$	$(34, 4.40 \times 10^{-7}, 0.55)$
PM1	$(17, 9.89 \times 10^{-7}, 0.29)$	$(17, 9.48 \times 10^{-7}, 0.30)$	$(16, 5.25 \times 10^{-7}, 0.28)$	$(19, 4.29 \times 10^{-7}, 0.33)$	$(21, 9.42 \times 10^{-7}, 0.35)$	$(19, 3.50 \times 10^{-7}, 0.30)$
PM2	$(15, 2.60 \times 10^{-7}, 0.28)$	$(15, 7.46 \times 10^{-7}, 0.26)$	$(15, 5.28 \times 10^{-7}, 0.26)$	$(18, 4.85 \times 10^{-7}, 0.29)$	$(17, 5.05 \times 10^{-7}, 0.29)$	$(15, 9.89 \times 10^{-7}, 0.23)$

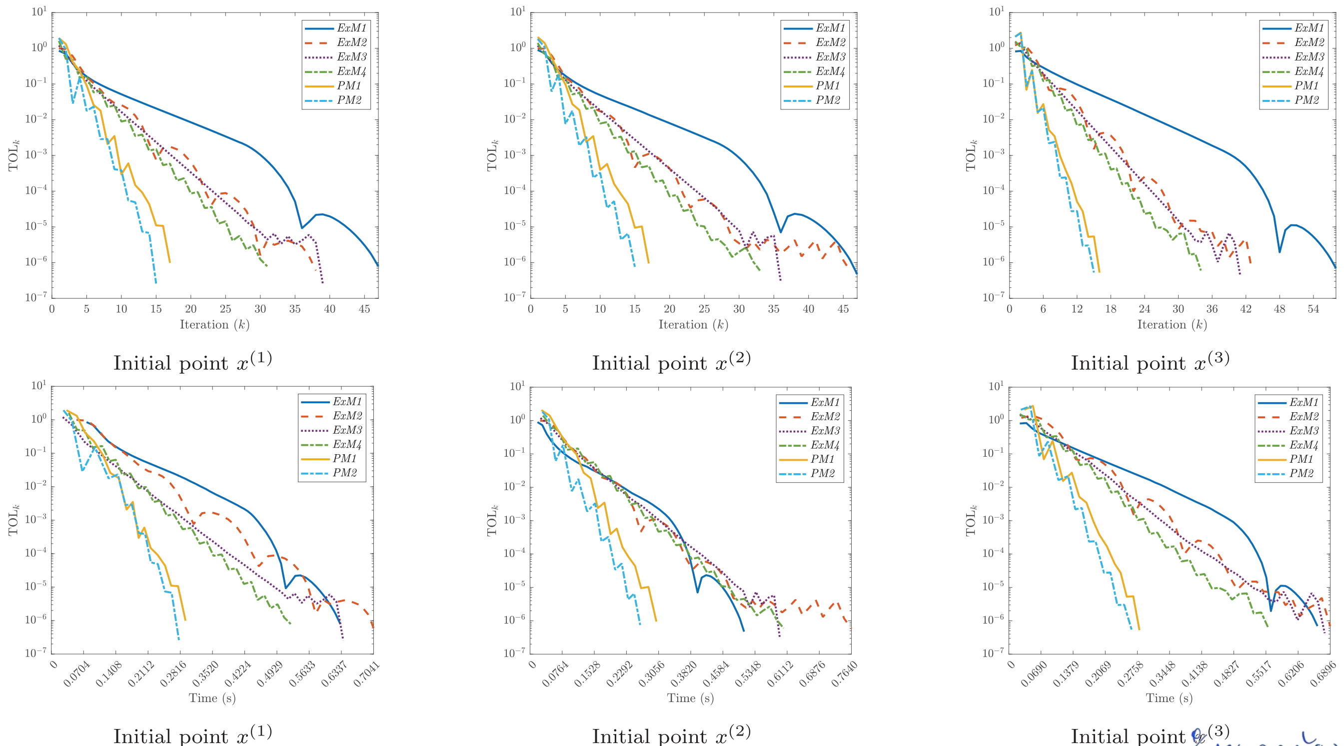

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FIGURE 1 | Residual norm decay for Example 4.1 with initial points $x^{(1)}$, $x^{(2)}$, and $x^{(3)}$. The top row plots the residual against iteration count, while the bottom row plots the residual against CPU time.

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1. The proposed algorithms (PM1 and PM2) consistently require the fewest iterations across all six initializations. For example, ExM1 frequently requires more than 50 iterations (e.g., 64 iterations for $x^{(6)}$), whereas both PM1 and PM2 converge in fewer than 21 iterations in every case. A similar pattern is observed for CPU time: PM2 typically finishes in under 0.30 s, whereas ExM1 and ExM3 often exceed 0.60 s, with ExM3 requiring nearly 1.0 s at $x^{(5)}$. Among the existing approaches, ExM4 is the most competitive, requiring approximately 30–34 iterations, but it remains noticeably slower than PM1 and PM2.

2. The existing methods exhibit significant sensitivity to the choice of initial point. For instance, ExM3 needs only 36 iterations at $x^{(2)}$ but as many as 63 iterations at $x^{(5)}$, with CPU time nearly doubling (0.59 s versus 0.95 s). In contrast, both proposed methods show remarkable robustness: each stays within the narrow range of 15–21 iterations for all six initial points, with consistently low CPU times (< 0.36 s). Additionally, the final residual errors for all methods

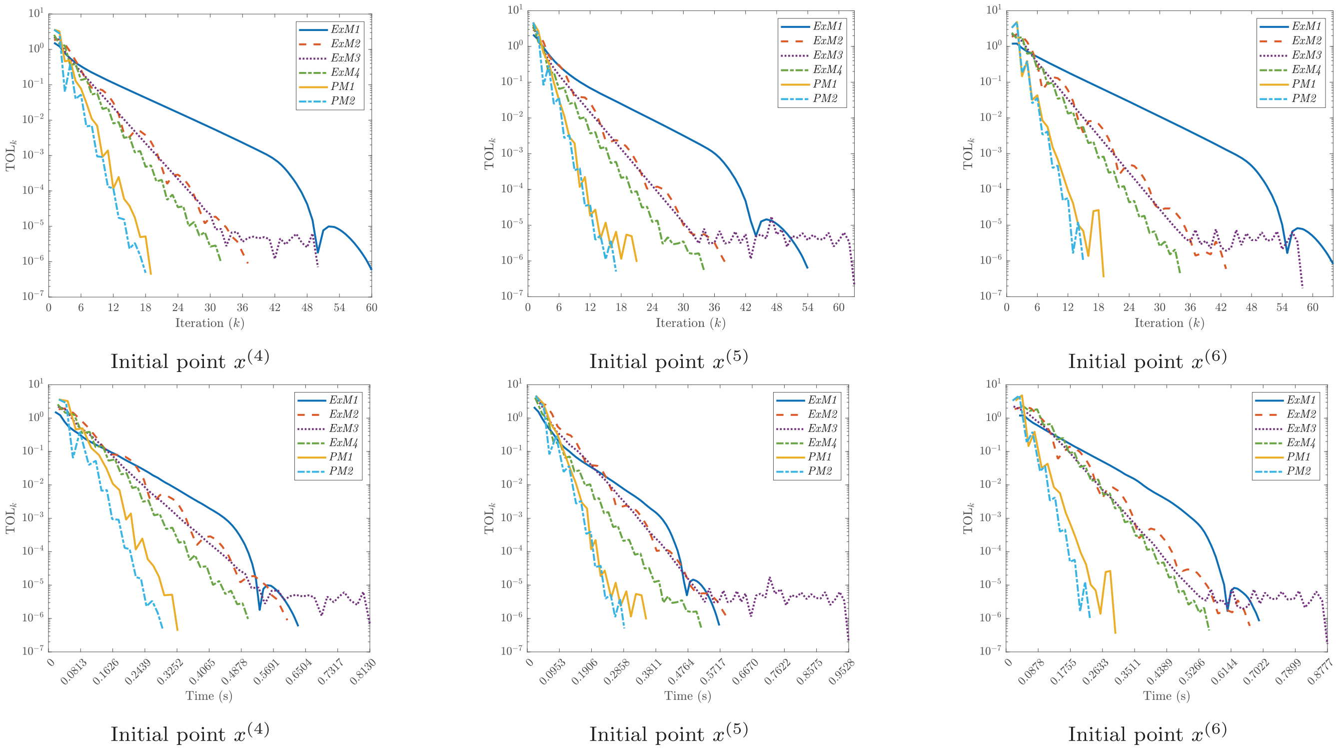

FIGURE 2 | Residual norm decay for Example 4.1 with initial points $x^{(4)}$, $x^{(5)}$, and $x^{(6)}$. The top row plots error versus iteration count, while the bottom row plots error versus CPU time.

are of comparable magnitude ($\mathcal{O}(10^{-7})$), demonstrating that the improved efficiency of PM1 and PM2 does not come at the cost of accuracy.

3. The proposed algorithms achieve substantial reductions in both iteration counts and execution times compared with existing methods, while also maintaining stable performance across diverse initializations. This combination of robustness, speed, and accuracy highlights the suitability of PM1 and PM2 for solving equilibrium models where both computational efficiency and reliability are essential. In summary, the experimental results confirm that the proposed methods provide notable improvements in efficiency and robustness, reinforcing their potential for broader application to large-scale and computationally demanding equilibrium problems.

Example 4.2. Let $\mathcal{H} = L^2([0,1])$ denote the Hilbert space of square-integrable real-valued functions on $[0,1]$, equipped with the inner product

$$\langle x, y\rangle = \int_0^1 x(t)\; y(t)\; dt, \qquad \forall\; x, y \in \mathcal{H},$$

and the corresponding norm

$$\|x\| = \left(\int_0^1 |x(t)|^2\; dt\right)^{1/2}.$$

The feasible set is the closed unit ball

$$C := \left\{\; x \in L^2([0,1]) \;\middle|\; \|x\| \le 1 \;\right\}.$$

Consider the operator $\mathcal{G} : C \to \mathcal{H}$ defined by

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$$\mathcal{G}(x)(t) = \int_0^1 (x(t) - H(t,s)\; f(x(s)))ds + g(t), \qquad t \in [0,1].$$

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Here the kernel H, the nonlinear mapping f, and the forcing term g are given by

$$H(t,s) = \frac{2ts\; e^{t+s}}{e\sqrt{e^2-1}}, \qquad f(x) = \cos(x), \qquad g(t) = \frac{2t\; e^t}{e\sqrt{e^2-1}}.$$

The associated bifunction $\mathcal{F} : \mathcal{C} \times \mathcal{C} \to \mathbb{R}$ is defined by

$$\mathcal{F}(x, y) := \langle \mathcal{G}(x),\ y - x \rangle.$$

It can be verified that the operator $\mathcal{G}$ is monotone and Lipschitz continuous on $\mathcal{C}$, with Lipschitz constant $L = 2$. In contrast with Example 4.1, this equilibrium problem is posed in an infinite-dimensional Hilbert space, offering a more challenging computational setting.

In this section, we present a series of numerical experiments to illustrate the applicability and effectiveness of the proposed methods in solving variational inequality problems over infinite-dimensional domains. Specifically, we consider Example 4.2, formulated in the Hilbert space $\mathcal{H} = L^2([0, 1])$. The aim of this experiment is to examine the convergence and stability of the algorithms when applied to a monotone and Lipschitz continuous operator $\mathcal{G}$ acting on an infinite-dimensional space. This framework highlights not only the robustness of the proposed approaches but also their potential applicability to functional analytical models.

To ensure a comprehensive evaluation, the algorithms are initialized with six distinct starting points in $\mathcal{H}$:

$$\begin{aligned} &x_0^{\{1\}}(t) = 0, && x_0^{\{2\}}(t) = 1, && x_0^{\{3\}}(t) = t, \\ &x_0^{\{4\}}(t) = \sin(\pi t), && x_0^{\{5\}}(t) = \cos(\pi t), && x_0^{\{6\}}(t) = e^{-t}, \qquad t \in [0, 1]. \end{aligned}$$

These initializations capture constant, linear, oscillatory, and exponentially decaying behaviors, thereby providing a meaningful basis for evaluating algorithmic performance under a broad range of functional configurations.

Observations and Analysis. From Table 3 and Figures 3 and 4, several noteworthy observations can be made:

1. Although all competing methods achieve final errors on the order of 10^{-10}, their computational efficiency varies significantly. Among the extragradient methods, *ExM1* is the slowest, requiring more than 150 iterations for certain initializations. By contrast, *ExM4* is the most efficient among the extragradient schemes, generally converging within 90–96 iterations and requiring nearly 1 s of CPU time.
2. The proposed projection-based methods (*PM1* and *PM2*) demonstrate superior numerical performance. In particular, *PM2* consistently achieves the fastest convergence, requiring only 27–47 iterations with CPU times in the range of 0.35–0.51 s.
3. The choice of initial point affects the numerical behavior of the algorithms. Constant initialization $x_0^{\{2\}}(t) = 1$ and linear initialization $x_0^{\{3\}}(t) = t$ generally lead to larger iteration counts and longer CPU times compared to oscillatory ($\sin(\pi t)$, $\cos(\pi t)$) or exponentially decaying (e^{-t}) initializations. Among the extragradient schemes, *ExM1* is particularly sensitive to the starting point, with iteration counts nearly doubling between $x_0^{\{1\}}(t) = 0$ and $x_0^{\{2\}}(t) = 1$. In contrast, the proposed projection-based methods exhibit remarkably stable performance across all initializations, with only minor variations in both iteration counts and execution times.

TABLE 3 | Numerical results for Example 4.2 in $L^2([0, 1])$ with six initial points $x_0^{\{i\}}$, $i = 1, \dots, 6$. Each entry reports (Iterations, Final Error, Time [s]). The comparison covers extragradient methods (ExM1–ExM4) and projection methods (PM1, PM2).

Method	$x_0^{\{1\}}(t) = 0$	$x_0^{\{2\}}(t) = 1$	$x_0^{\{3\}}(t) = t$	$x_0^{\{4\}}(t) = \sin(\pi t)$	$x_0^{\{5\}}(t) = \cos(\pi t)$	$x_0^{\{6\}}(t) = e^{-t}$
ExM1	(78, 8.48×10^{-10}, 0.553)	(155, 8.91×10^{-10}, 1.572)	(149, 9.37×10^{-10}, 2.122)	(152, 8.93×10^{-10}, 1.546)	(153, 9.52×10^{-10}, 1.482)	(151, 9.79×10^{-10}, 1.490)
ExM2	(71, 9.98×10^{-10}, 1.100)	(117, 8.62×10^{-10}, 1.221)	(114, 8.96×10^{-10}, 1.190)	(115, 9.37×10^{-10}, 1.204)	(116, 8.38×10^{-10}, 2.179)	(115, 8.84×10^{-10}, 1.140)
ExM3	(69, 8.96×10^{-10}, 0.740)	(113, 8.33×10^{-10}, 1.131)	(107, 8.30×10^{-10}, 1.121)	(109, 9.56×10^{-10}, 1.128)	(111, 9.36×10^{-10}, 1.112)	(113, 8.41×10^{-10}, 1.153)
ExM4	(58, 8.97×10^{-10}, 0.533)	(96, 6.64×10^{-10}, 0.965)	(90, 8.42×10^{-10}, 0.954)	(92, 8.87×10^{-10}, 0.982)	(94, 8.14×10^{-10}, 0.945)	(96, 6.72×10^{-10}, 0.984)
PM1	(38, 7.18×10^{-10}, 0.346)	(62, 9.71×10^{-10}, 0.680)	(59, 8.05×10^{-10}, 0.630)	(61, 7.02×10^{-10}, 0.659)	(61, 9.88×10^{-10}, 0.598)	(62, 9.66×10^{-10}, 0.651)
PM2	(27, 7.24×10^{-10}, 0.345)	(47, 5.31×10^{-10}, 0.493)	(43, 9.56×10^{-10}, 0.479)	(45, 6.96×10^{-10}, 0.500)	(45, 9.45×10^{-10}, 0.480)	(47, 5.90×10^{-10}, 0.513)

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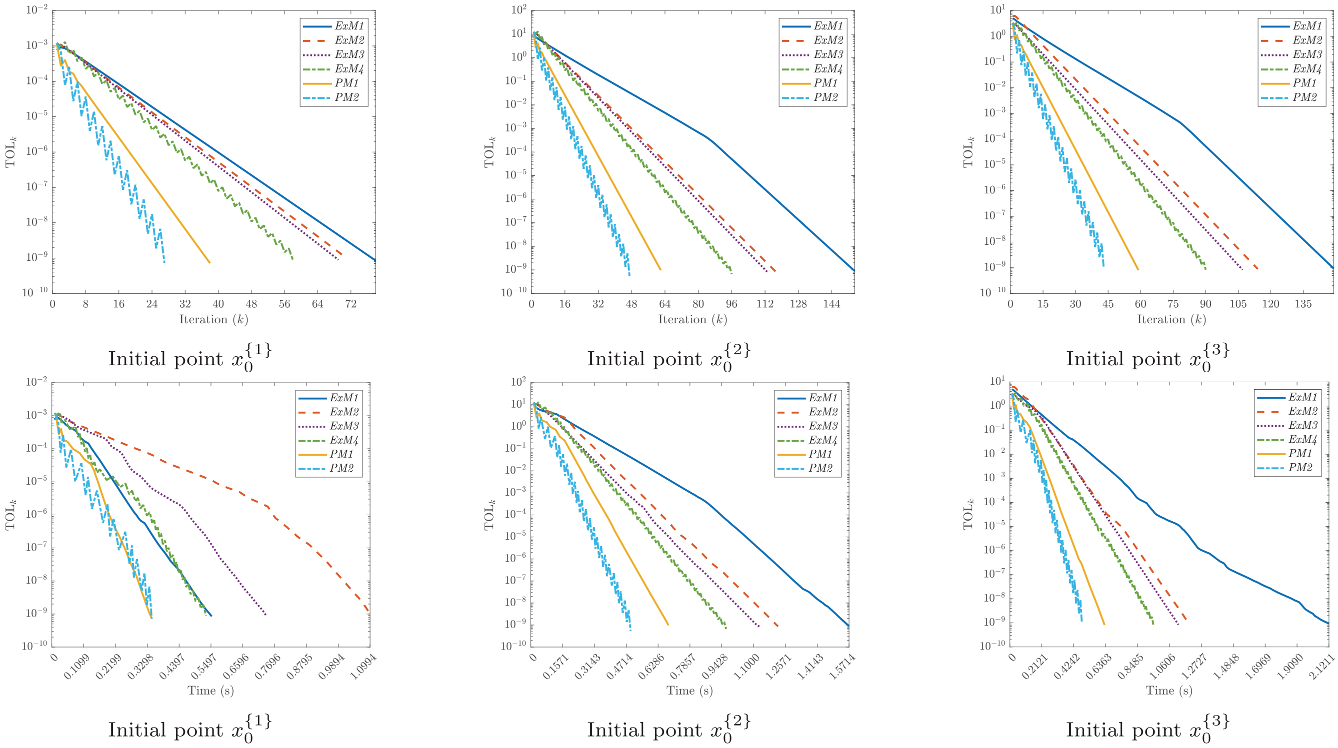

FIGURE 3 | Residual norm decay for Example 4.2 with initial points $x_0^{\{1\}}$, $x_0^{\{2\}}$, and $x_0^{\{3\}}$. Top: error vs. iterations. Bottom: Error vs. CPU time. The plots compare extragradient methods (ExM1 – ExM4) with the proposed projection-based methods (PM1, PM2).

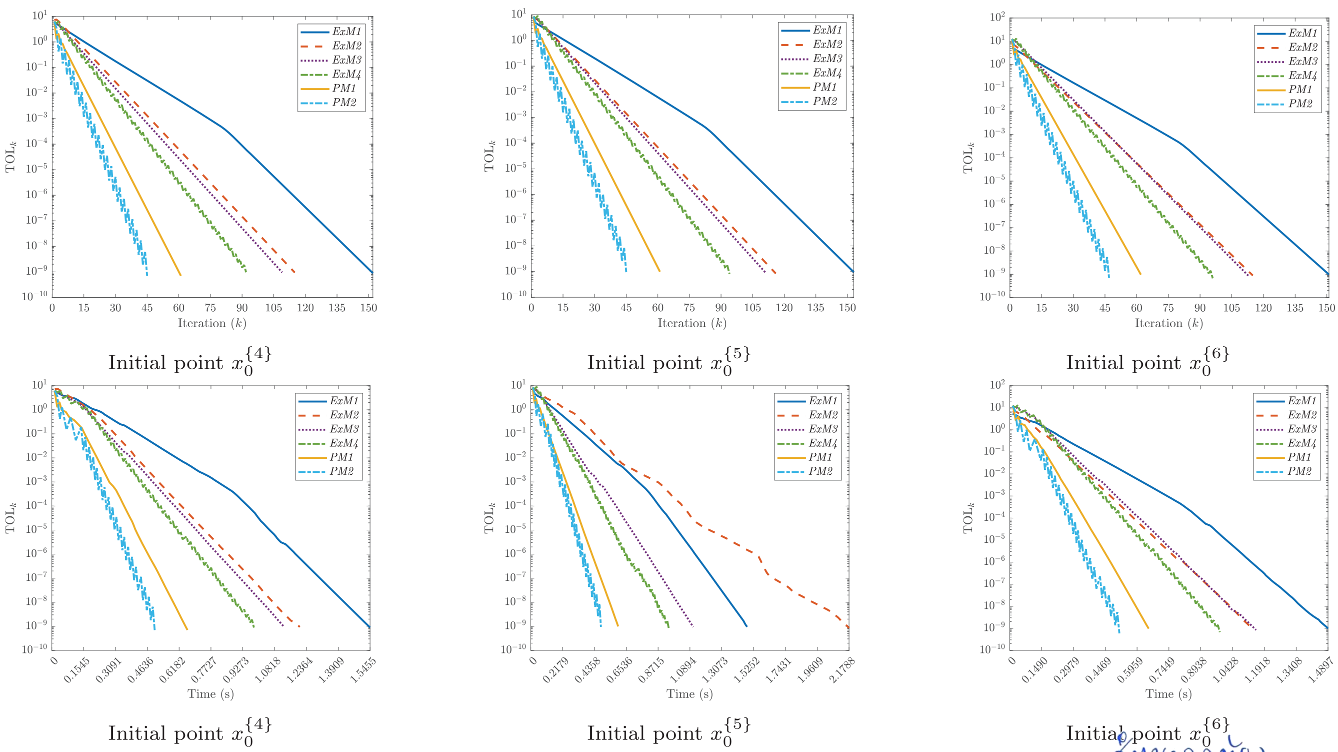

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FIGURE 4 | Residual norm decay for Example 4.2 with initializations $x_0^{\{4\}}, x_0^{\{5\}}, x_0^{\{6\}}$. Top: error vs. iterations. Bottom: Error vs. CPU time. The results illustrate the effect of oscillatory and exponentially decaying initializations on the convergence behavior of ExM1 – ExM4 and PM1 – PM2.

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Overall, *PM2* emerges as the most effective method for this example, combining robustness, accuracy, and computational efficiency. It not only outperforms all extragradient algorithms but also provides consistently stable performance across diverse initializations in the infinite-dimensional setting.

Example 4.3. In this example, we illustrate the effectiveness of the proposed methods in an image restoration problem. Each image is represented by $D := M \times N$ pixels, where the intensity of each pixel lies in the interval $[0, 255]$, corresponding to standard grayscale values. The reconstruction problem is formulated in the real Hilbert space $\mathbb{R}^D$ equipped with the Euclidean norm $\|\cdot\|$. The feasible set of pixel values is therefore

$$C := [0, 255]^D.$$

Let $\overline{x}$ denote the original (unknown) image and let y^* denote the observed degraded image. The degradation process is modeled as

$$y^* = A\overline{x} + \xi,$$

where A is a blurring operator (typically represented by a point spread function or convolution matrix) and ξ is an additive noise term. The objective is to reconstruct an approximation of $\overline{x}$ from the observed data y^* using the known forward operator A. This leads to the constrained optimization problem

$$\min_{x \in C} \frac{1}{2}\|Ax - y^*\|^2,$$

where x is the candidate restored image. The objective function is given by

$$\varphi(x) := \frac{1}{2}\|Ax - y^*\|^2,$$

which measures the discrepancy between the degraded image and its reconstruction. Since A is linear and $\|\cdot\|^2$ is convex, the function φ is convex. Consequently, the problem can be equivalently formulated as an equilibrium problem with the bifunction

$$\mathcal{F}(x, y) := \varphi(y) - \varphi(x), \qquad \forall\, x, y \in C.$$

Image Quality Metrics. To evaluate the quality of the restored image x, we employ three widely used image-quality measures:

TABLE 4 | Numerical results under a Gaussian blur model (kernel size 9×9, $\sigma = 1.6$). Each row reports the outcome after 300 iterations for the four extragradient methods (ExM1–ExM4) and the two projection-based methods (PM1, PM2). Metrics include runtime, SNR/PSNR/SSIM, and pixel-wise calibration statistics on the Y channel.

Method	Iter	Time (s)	SNR (dB)	PSNR (dB)	SSIM	r	R^2	Slope	Intercept	RMSE	MAE	Bias
ExM1	300	1.0904	17.66	23.24	0.4414	0.9604	0.9224	0.9619	4.53	17.55	13.19	0.00
ExM2	300	1.4609	17.99	23.57	0.4621	0.9631	0.9276	0.9592	4.86	16.90	12.54	0.02
ExM3	300	1.4270	18.33	23.91	0.4938	0.9658	0.9327	0.9589	4.90	16.26	11.91	0.02
ExM4	300	1.5298	18.84	24.42	0.5450	0.9694	0.9398	0.9564	5.22	15.33	10.79	0.03
PM1	300	1.4622	19.15	24.73	0.5912	0.9715	0.9438	0.9537	5.48	14.79	10.02	−0.02
PM2	300	1.4326	19.32	24.90	0.6758	0.9726	0.9459	0.9469	6.33	14.50	8.98	0.02

Note: **Iter**: Number of iterations (here fixed at 300).
Time (s): Execution time of each method.
SNR (dB): Signal-to-noise ratio of the restored image; higher values indicate better fidelity.
PSNR (dB): Peak signal-to-noise ratio, a standard metric for image quality.
SSIM: Structural similarity index (range $[0, 1]$); higher values indicate closer structural similarity to the reference image.
r: Linear correlation coefficient between ground truth and restored luminance values.
R^2: Coefficient of determination, measuring how well restored values approximate the ground truth.
Slope/Intercept: Parameters of the regression line (restored vs. ground truth luminance).
RMSE: Root mean square error.
MAE: Mean absolute error.
Bias: Average deviation of restored luminance from the ground truth.

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1. **Signal-to-Noise Ratio (SNR):**

$$\text{SNR} := 20\log_{10}\left(\frac{\|\overline{x}\|}{\|\overline{x}-x\|}\right).$$

   SNR quantifies the strength of the true image relative to the reconstruction error, serving as a numerical indicator of error decay.

2. **Peak Signal-to-Noise Ratio (PSNR):**

$$\text{PSNR} := 10\log_{10}\left(\frac{255^2}{1D\|\overline{x}-x\|^2}\right).$$

   PSNR compares the maximum possible pixel intensity (255) with the mean squared error between $\overline{x}$ and x. It is the most widely used performance benchmark in the image-restoration literature.

3. **Structural Similarity Index (SSIM):** SSIM assesses perceptual similarity by jointly comparing luminance, contrast, and structural information between $\overline{x}$ and x. Its values lie in $[0,1]$, with 1 indicating perfect structural agreement. SSIM therefore provides a perceptually meaningful evaluation beyond pixel-wise errors.

These three measures together capture complementary aspects of reconstruction quality: numerical accuracy (SNR), standard benchmark performance (PSNR), and perceptual fidelity (SSIM).

Initialization. For the experiments, we initialize the algorithm with

$$x_0 = \mathbf{1} \in \mathbb{R}^D, \qquad x_1 = \mathbf{0} \in \mathbb{R}^D,$$

corresponding to constant images whose pixel values are all one and zero, respectively. These choices offer simple yet effective starting points for the iterative method.

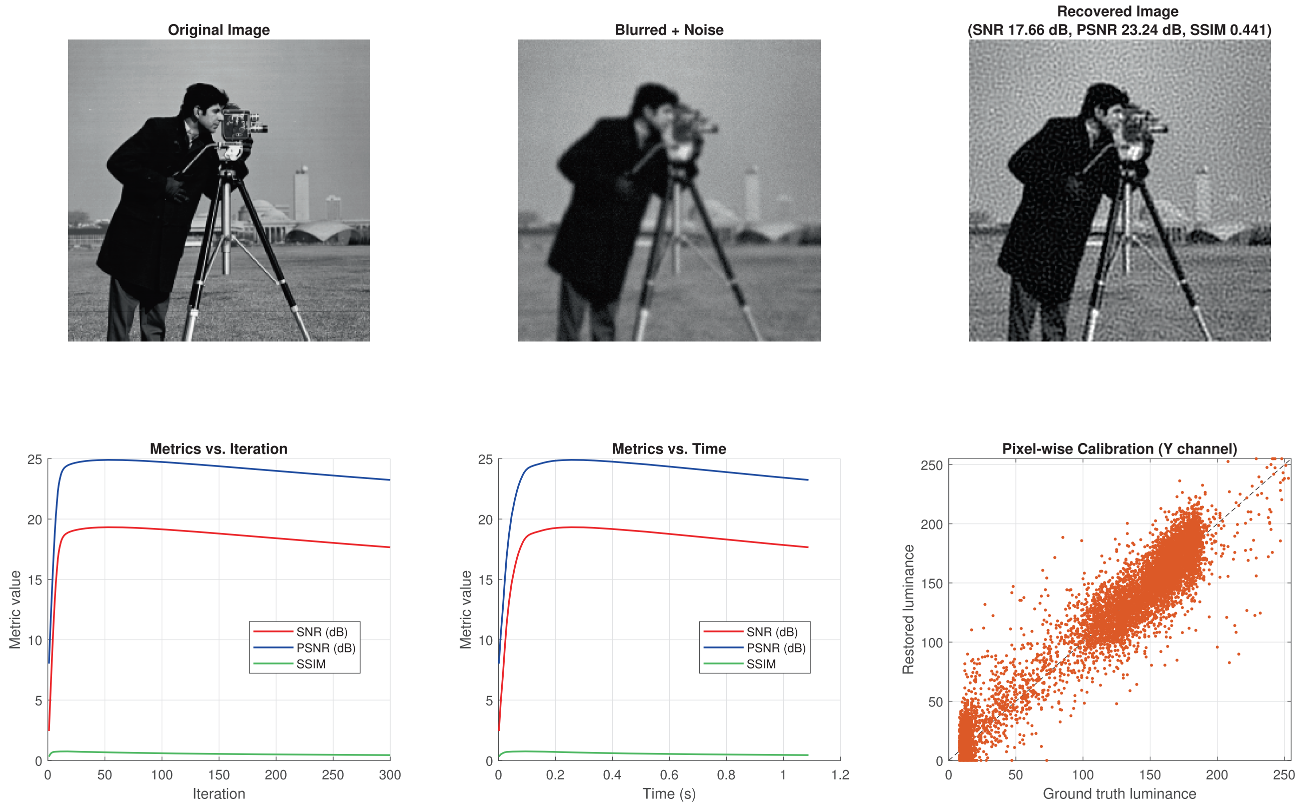

FIGURE 5 | Performance of method *ExM1* on Example 4.3 under Gaussian blur (9×9, $\sigma = 1.6$). Top row: (a) Original image, (b) blurred and noisy observation, and (c) restored image with quality metrics (SNR, PSNR, SSIM). Bottom row: (d) evolution of metrics with respect to iterations, (e) evolution of metrics with respect to runtime, and (f) pixel-wise calibration of restored versus ground truth luminance.

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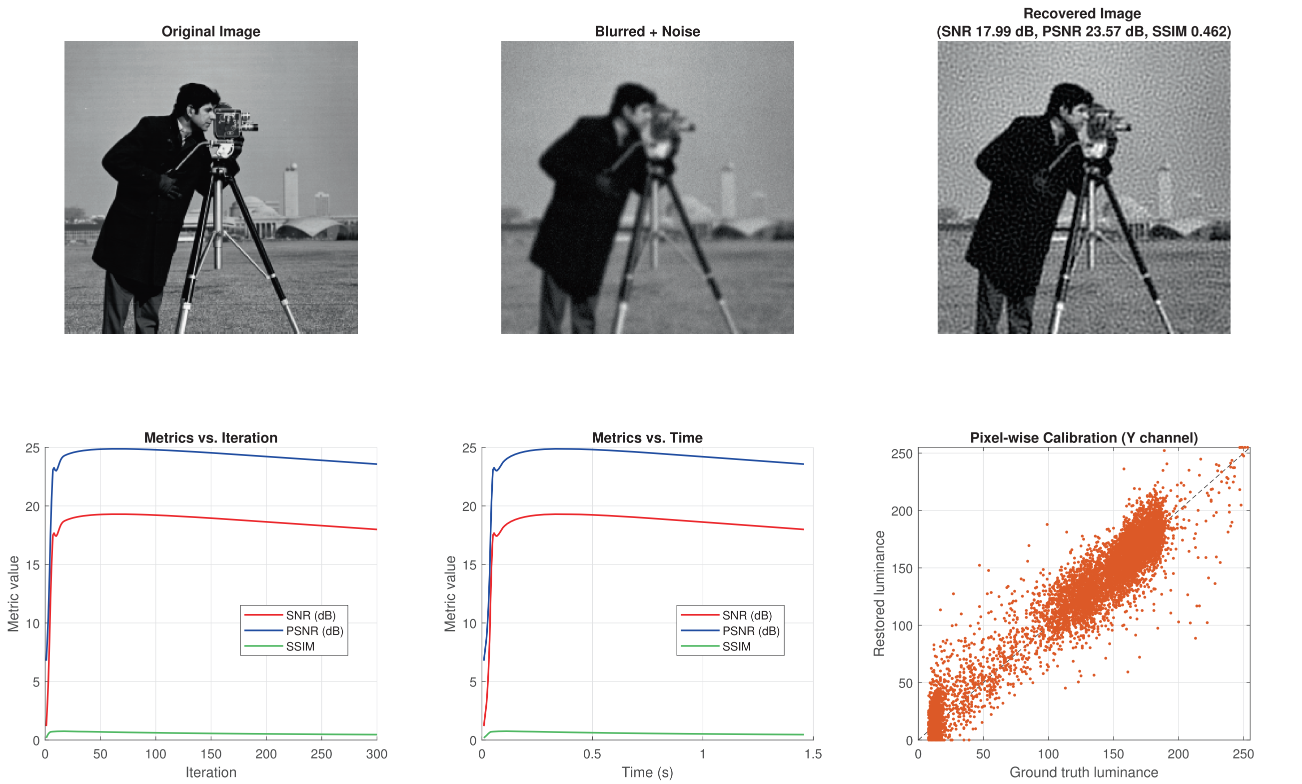

FIGURE 6 | Performance of method *ExM2* on Example 4.3 under Gaussian blur (9×9, $\sigma = 1.6$). Subfigures: (a) Original image, (b) blurred and noisy observation, (c) restored image, (d) evolution of metrics across iterations, (e) evolution of metrics over runtime, and (f) pixel-wise calibration.

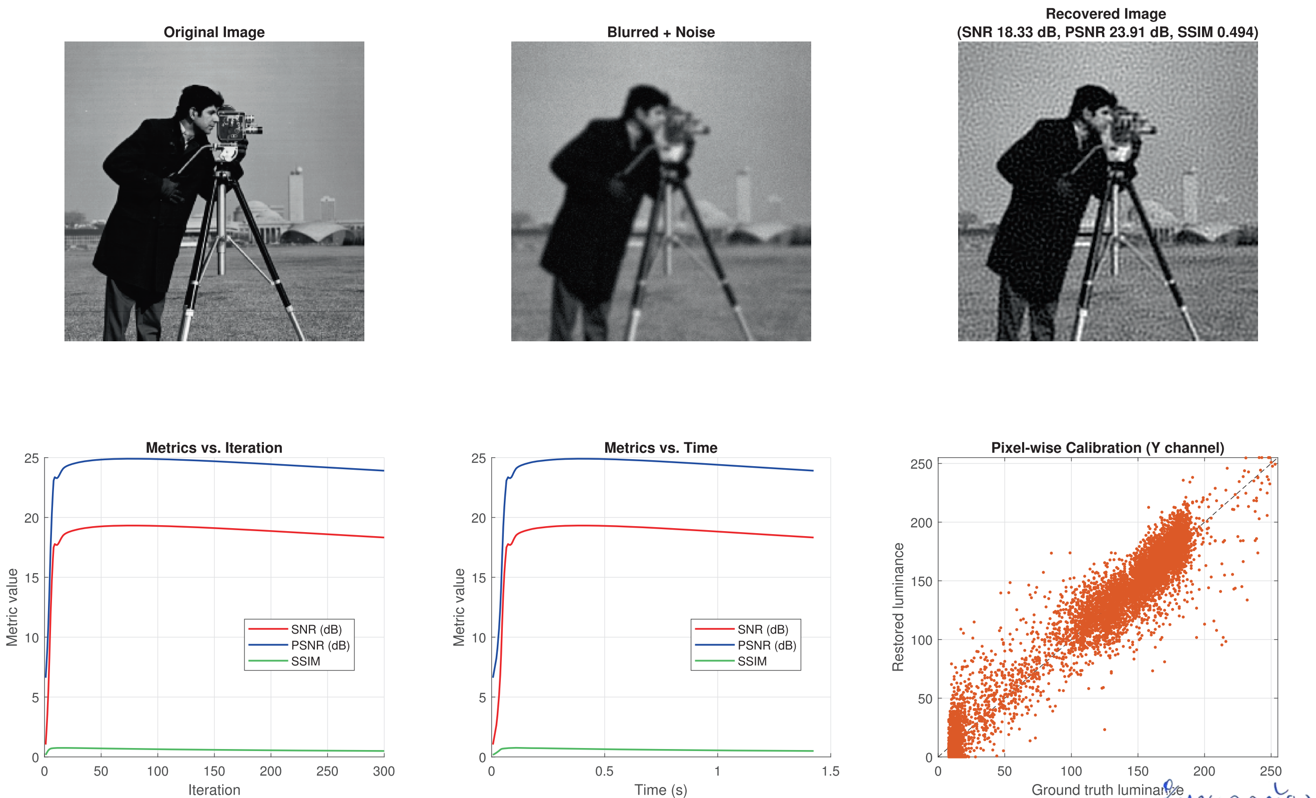

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FIGURE 7 | Performance of method *ExM3* on Example 4.3 under Gaussian blur (9×9, $\sigma = 1.6$). Subfigures: (a) Original image, (b) blurred and noisy observation, (c) restored image, (d) metric progression across iterations, (e) metric progression over runtime, and (f) pixel-wise calibration.

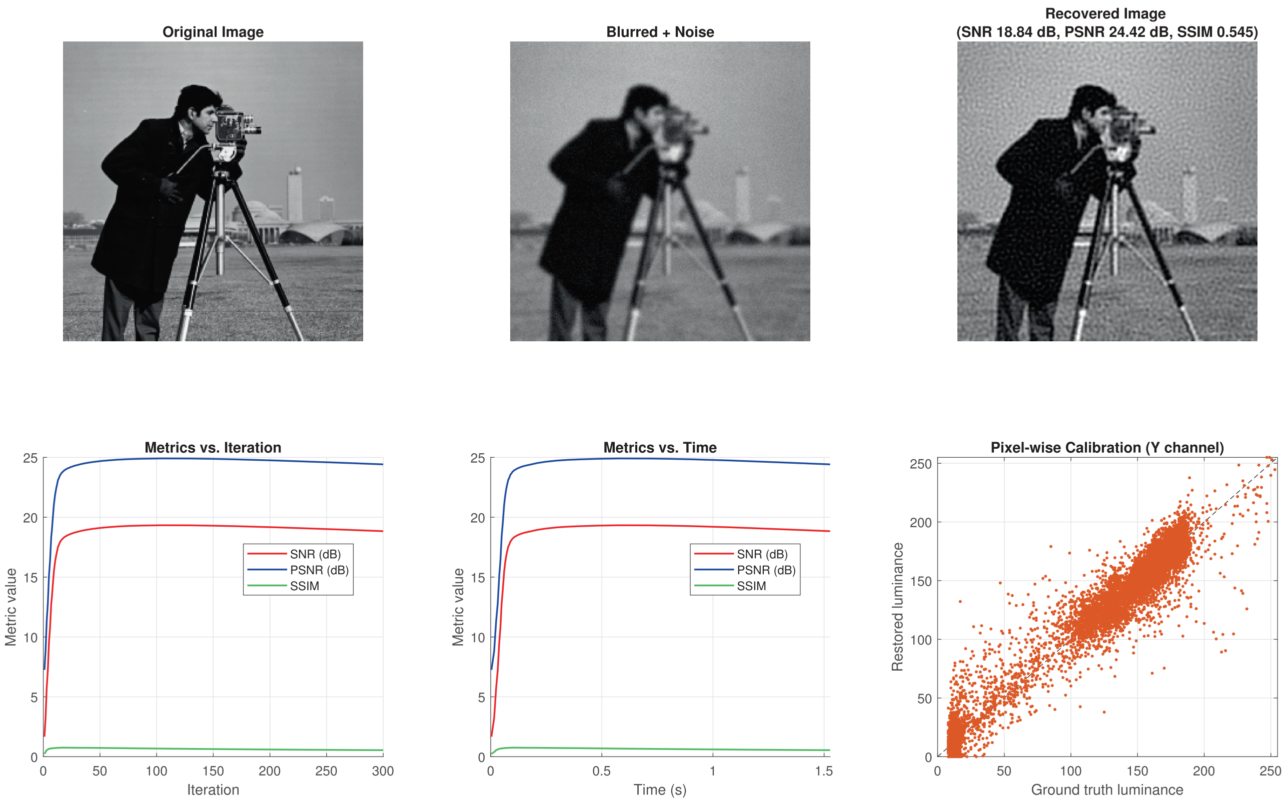

FIGURE 8 | Performance of method *ExM4* on Example 4.3 under Gaussian blur (9×9, $\sigma = 1.6$). Subfigures: (a) Original image, (b) degraded observation, (c) restored image, (d) SNR/PSNR/SSIM versus iteration, (e) SNR/PSNR/SSIM versus runtime, and (f) pixel-wise calibration.

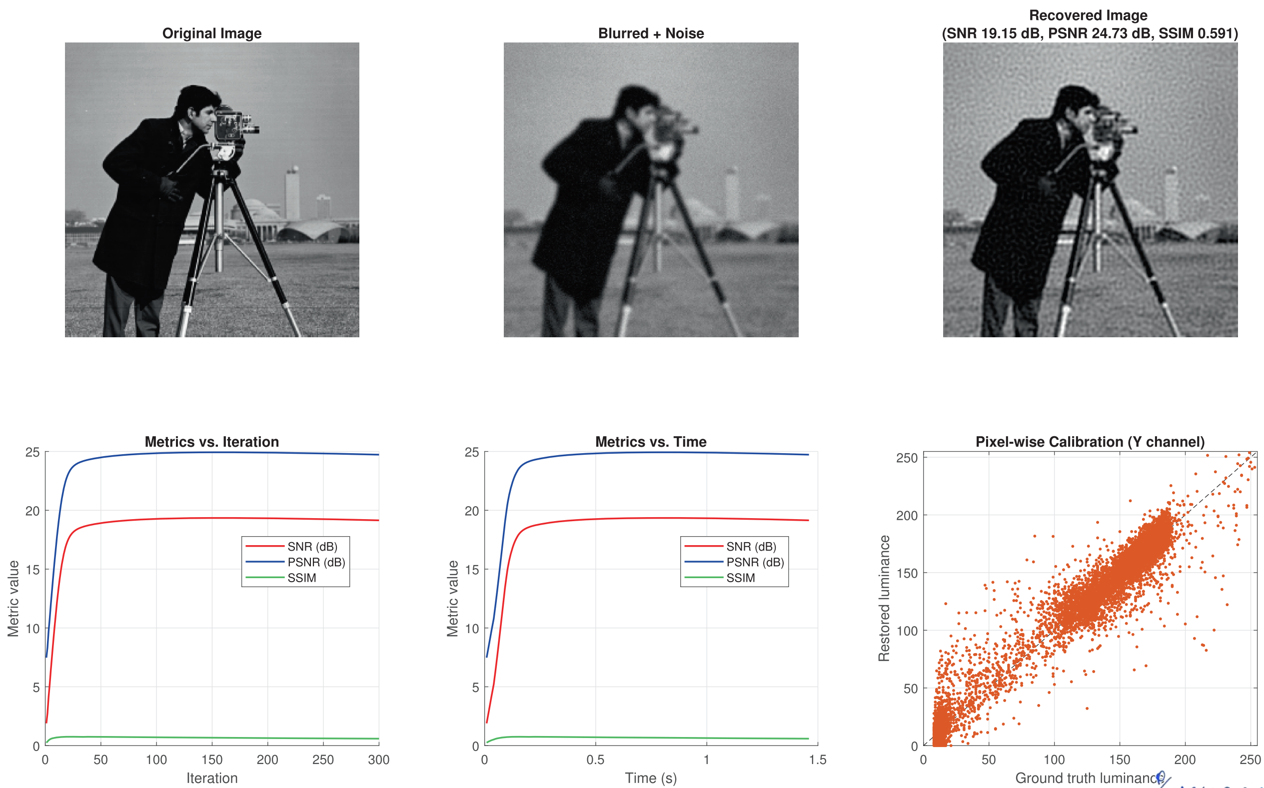

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FIGURE 9 | Performance of proposed method *PM1* on Example 4.3 under Gaussian blur (9×9, $\sigma = 1.6$). Subfigures: (a) Original image, (b) blurred and noisy observation, (c) restored image, (d) evolution of metrics with respect to iterations, (e) evolution of metrics with respect to runtime, and (f) pixel-wise calibration.

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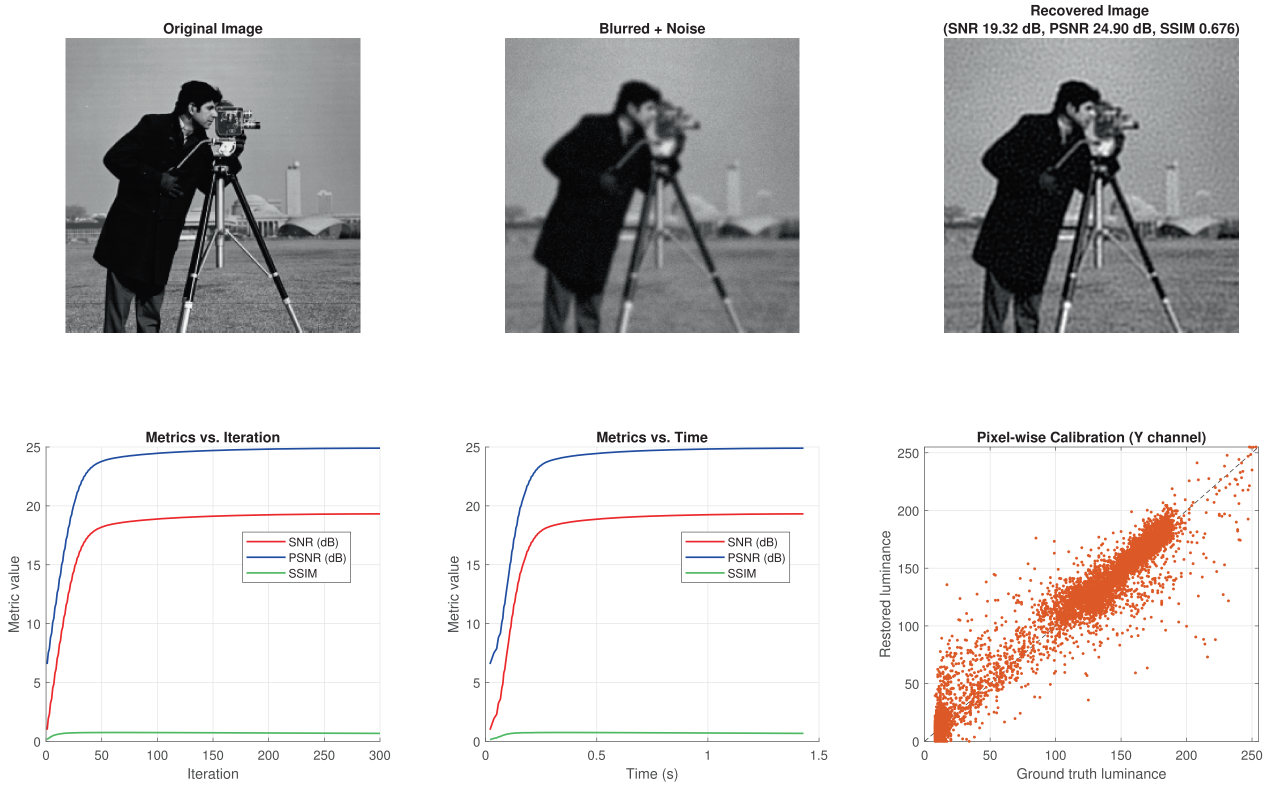

FIGURE 10 | Performance of proposed method *PM2* on Example 4.3 under Gaussian blur (9×9, $\sigma = 1.6$). Subfigures: (a) Original image, (b) blurred and noisy observation, (c) restored image with quality metrics, (d) convergence curves versus iteration, (e) convergence curves versus runtime, and (f) pixel-wise calibration.

Observations. From the numerical results in Table 4 and Figures 5–10, we obtain the following observations:

1. All methods were executed for a fixed number of 300 iterations. The execution times are comparable, ranging from 1.09 s (ExM1) to 1.53 s (ExM4). Importantly, the projection-based methods (PM1 and PM2) do not introduce additional computational overhead compared with the extragradient schemes.
2. Restoration quality improves progressively across the methods. Specifically, the SNR increases from 17.66 dB (ExM1) to 19.32 dB (PM2). A similar trend holds for PSNR, which rises from 23.24 dB (ExM1) to 24.90 dB (PM2). The most pronounced gain is observed in SSIM: while ExM1 achieves only 0.4414, PM2 attains 0.6758, indicating substantially stronger structural similarity with the reference image.
3. Correlation coefficients remain consistently high ($r \geq 0.96$, $R^2 \geq 0.92$), confirming strong agreement between the ground truth and the restored luminance values. The regression slope decreases slightly from 0.9619 (ExM1) to 0.9469 (PM2), while the intercept increases, indicating a minor brightness bias. Error measures also improve steadily: ExM1 yields the largest errors (RMSE 17.55, MAE 13.19), whereas PM2 achieves the smallest (RMSE 14.50, MAE 8.98). The bias remains negligible across all methods, with values between −0.02 and 0.03.
4. Overall, performance improves progressively from ExM1 through ExM4, with the projection-based methods (PM1 and PM2) delivering the best restoration quality. In particular, *PM2* provides the most favorable balance between perceptual similarity (SSIM) and numerical fidelity (SNR, PSNR, RMSE, MAE), making it the most effective method in this experiment.

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Author Contributions

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Habib ur Rehman: writing – original draft; writing – review and editing; conceptualization; methodology; investigation; formal analysis; software; project administration. **Nattawut Pholasa:** writing – original draft; writing – review and editing; funding acquisition; validation; supervision; project administration. **Nuttapol Pakkaranang:** writing – original draft; writing – review and editing; conceptualization; methodology; investigation; formal analysis; software; project administration; visualization; supervision; validation.

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Conflicts of Interest

The authors declare no conflicts of interest.

Data Availability Statement

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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