Scopus Preview

Source details

Journal of Nonlinear and Convex Analysis

Years currently covered by Scopus: from 2010 to 2025

Publisher: Yokohama Publishers

ISSN: 1345-4773 E-ISSN: 1880-5221

Subject area: Mathematics: Geometry and Topology | Mathematics: Analysis | Mathematics: Control and Optimization | Mathematics: Applied Mathematics

Source type: Journal

View all documents > Set document alert Save to source list

CiteScore 2024
1.6

SJR 2024
0.310

SNIP 2024
0.459

CiteScore CiteScore rank & trend Scopus content coverage

CiteScore 2024

$$1.6 = \frac{\text{1,107 Citations 2021 - 2024}}{\text{712 Documents 2021 - 2024}}$$

Calculated on 05 May, 2025

CiteScoreTracker 2025

$$0.7 = \frac{\text{488 Citations to date}}{\text{654 Documents to date}}$$

Last updated on 05 December, 2025 • Updated monthly

CiteScore rank 2024

Category	Rank	Percentile
Mathematics — Geometry and Topology	#46/111	59th
Mathematics — Analysis	#111/198	44th
Mathematics		

View CiteScore methodology > CiteScore FAQ > Add CiteScore to your site

SI EPI SCImago

SCImago Journal & Country Rank

Journal Rankings | Journal Value | Country Rankings | Viz Tools | About Us | Search

Journal of Nonlinear and Convex Analysis

Japan | Universities and research institutions | Media Ranking

Country

SCIMAGO INSTITUTIONS RANKINGS

SCImago Media Rankings

Subject Area and Category

Mathematics
- Analysis
- Applied Mathematics
- Control and Optimization
- Geometry and Topology

Publisher

Yokohama Publishers

SJR 2024

0.310 Q3

H-Index

39

Publication type

Journals

ISSN

13454773, 18805221

Coverage

2010-2024

Information

Home ↗

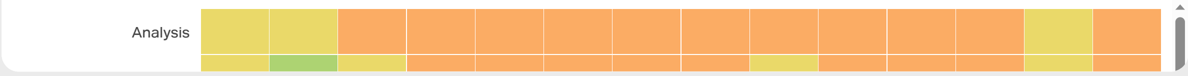
How to publish in this journal ↗

tlau@math.ualberta.ca ↗

Scope

Journal of Nonlinear and Convex Analysis will publish carefully selected mathematical papers in nonlinear analysis and convexity theory. Papers submitted treating various aspectos of nonlinear analysis, fixed point theory, game theory, optimization, analysis of set-valued mappings, partial differential equations, convexity and applications to economics, mathematics of finance and engineering are particularly encouraged.

Quartiles



Find similar journals

All quartiles | All countries | All subject categories | Clear filters | Download

Only Open Access Journals

1 - Journal of Nonlinear and Variational Analysis

64%

similarity

2 - Journal of Nonlinear Functional Analysis

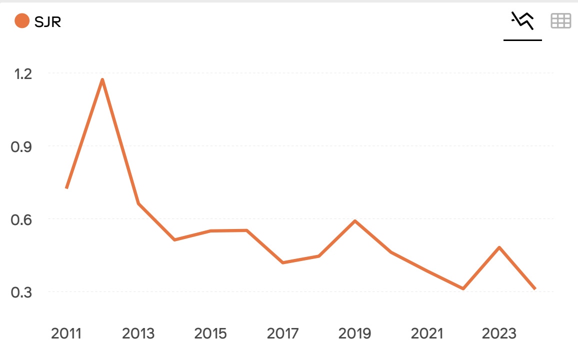
63%

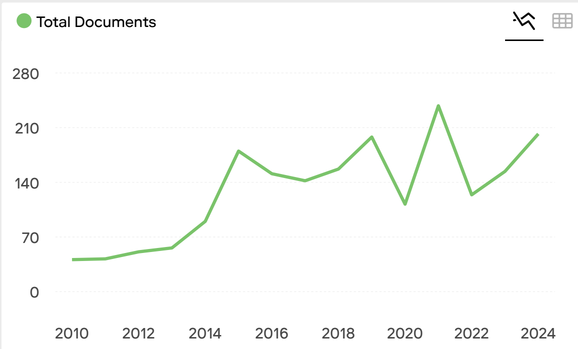
similarity

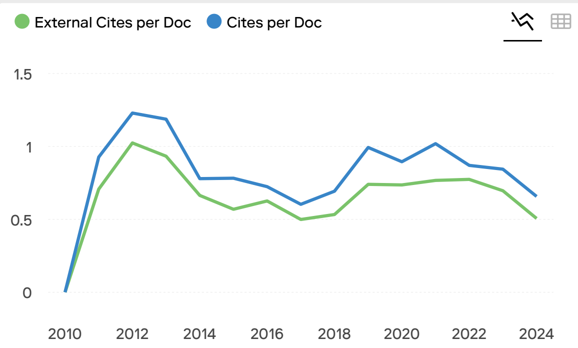
3 - Fixed Point Theory

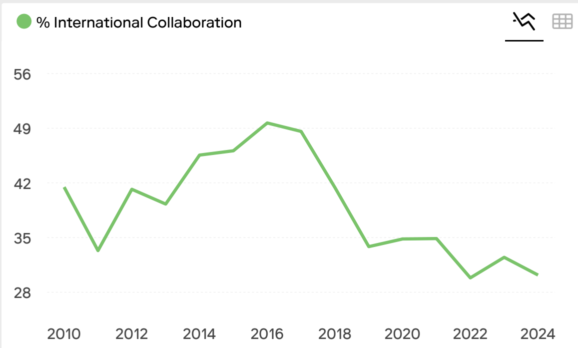
57%

similarity





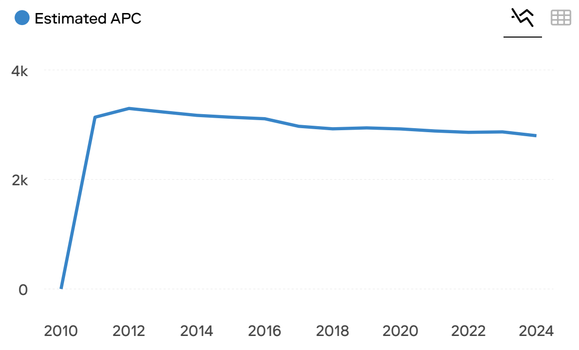


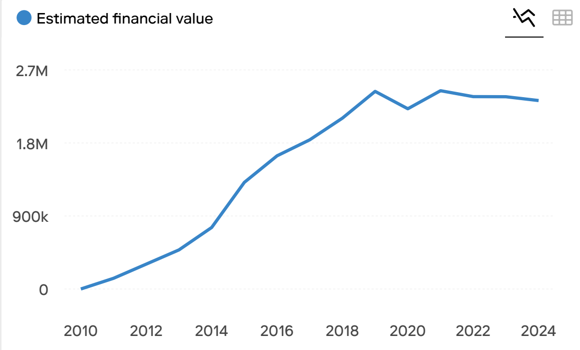


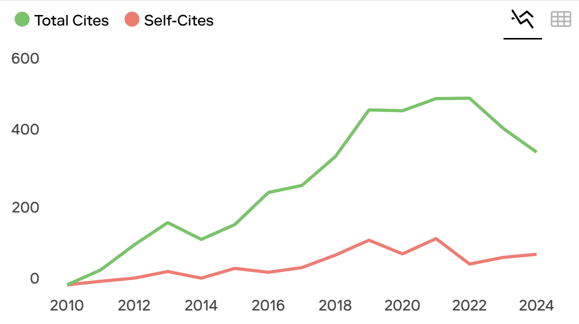
Cited documents | Uncited documents

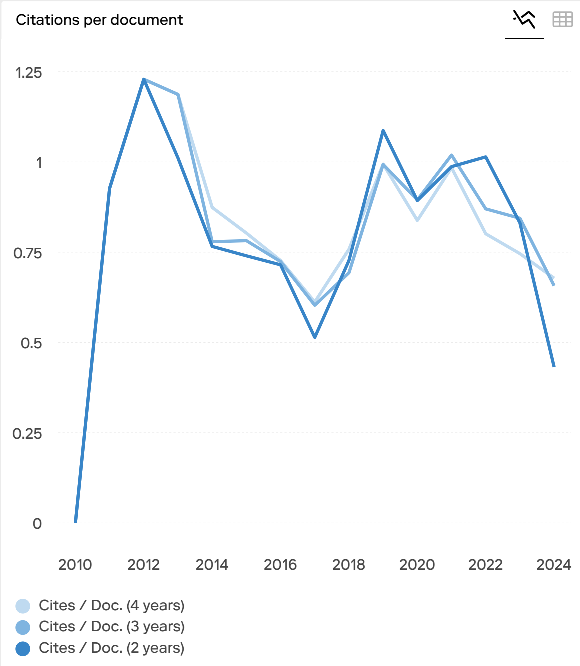
% Female Authors

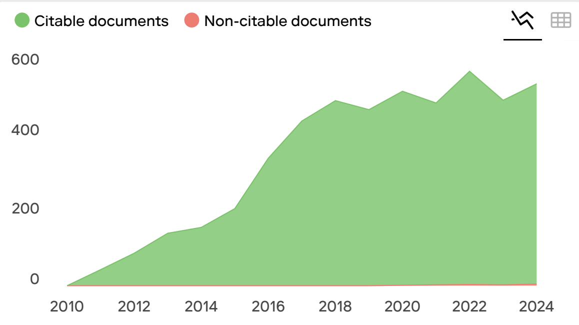
600

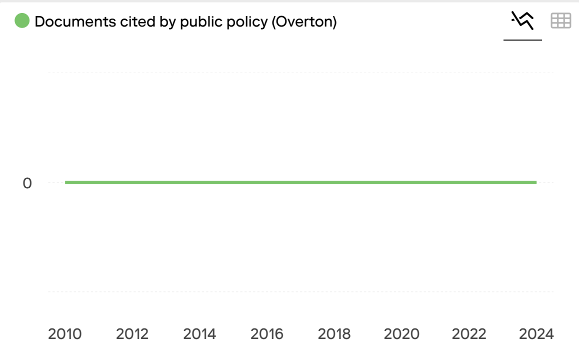


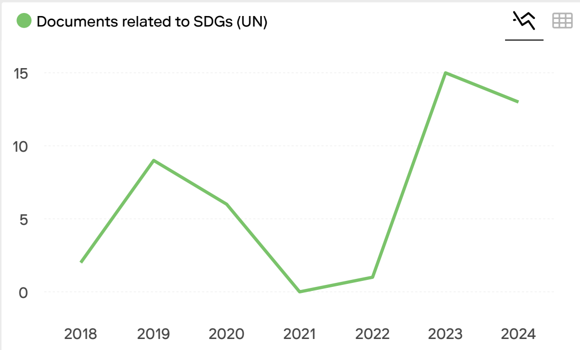












← Show this widget in your own website

Just copy the code below and paste within your html code:

<a href="https://www.scimag

Nuttapol Pakkaranang <nuttapol.pak@pcru.ac.th>

Special Issue of JNCA- Decision on Your Manuscript-Unified Approaches to Equilibrium and Fixed-Point Problems: An Improved Viscosity-Based Subgradient Extragradient Framework

SIofJNCA2025 <specialissuejnca2025@gmail.com> 24 ตุลาคม 2568 เวลา 11:19
ถึง: Nuttapol Pakkaranang <nuttapol.pak@pcru.ac.th>

Dear Assistant Professor Nuttapol Pakkaranang, Ph.D.,

We are pleased to inform you that after careful review and consideration, your manuscript entitled
"Unified Approaches to Equilibrium and Fixed-Point Problems: An Improved Viscosity-Based Subgradient Extragradient Framework"

has been accepted for publication in the Special Issue: Fixed Point Theory and Convex Optimization with Applications on behalf of the contribution and honor for celebrating the 65th birthday of Professor Suthep Suantai, of the Journal of Nonlinear and Convex Analysis.

We are confident it will be of great interest to our readership. Your manuscript will now proceed to the production stage, where it will undergo copyediting and typesetting.
Please send your accepted manuscript .tex file by replying to this email within October 25, 2025.
You will be contacted by the production office with proofs for your review prior to final publication.

Important Note:

"If your article is accepted please change your tex file to the style of JNCA (You can find the style in sample file and please use jnca-sample.file). If you did not use JNCA style file or we add professional edits to your tex files, you should pay the following editing page charge"
(see http://www.yokohamapublishers.jp/JNCA-pagecharge-2020.pdf)

For more details, see http://www.ybook.co.jp/jnca.html

On behalf of the editorial team, we would like to congratulate you and your co-authors on this achievement and thank you for choosing the Journal of Nonlinear and Convex Analysis as the venue for your research. We look forward to seeing your article published soon.

Best regards,
Editorial Team
Assoc. Prof. Dr. Attapol Kaewkhao,
Assoc. Prof. Dr. Warunun Inthakon,
Assoc. Prof. Dr. Bancha Panyanak,
Assoc. Prof. Dr. Prasit Cholamjiak,
Assoc. Prof. Dr. Watcharaporn Cholamjiak

สำเนาถูกต้อง

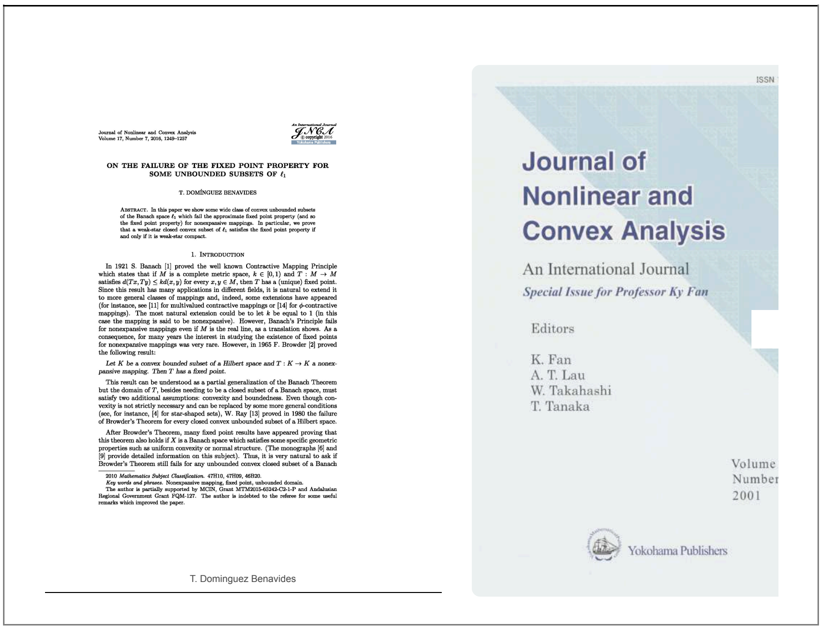
(นายณัฐพล ปักการะนัง)

An International Journal

Journal of Nonlinear and Convex Analysis

PRINT ISSN 1345-4773 ONLINE ISSN 1880-5221

EDITORS/EDITORIAL BOARD   EDITORIAL POLICY   FOR AUTHORS   ORDER FORM   PRICE LIST



NEW

****

Call for Papers (Deadline 17 October):
Special issue in Honor of Prof. Hong-Kun Xu's 65th Birthday

Call for Papers (Deadline 31 July):
Fixed Point Theory and Convex Optimization with Applications
On behalf of the contribution and honor for celebration the 65th birthday of
Professor Suthep Suantai

****

Issue 6 & 7, 2025
Fussy Systems and Data Mining

Issue 5, 2025
Dedicated to Professor Sehie Park on the occasion of his 90th birthday

Issue 4, 2025
ICAA-24, JMI India

Issue 3, 2025 (Nonlinear and Variational Analysis 2024)
Dedicated to Prof. Shih-Sen Chang on the occasion of his 91th birthday

Issue 12, 2024 (In honor of Professor Do Sang Kim on the occasion of his 70th birthday)

Issue 11

Call for Papers: Fixed Point Theory and Applications to
Fractional Ordinary and Partial Difference and Differential Equations

Issue 9

Nonlinear and Variation Analysis 2023
Issue 8

IC-Fixed Point Theory and Applicaton 2023
Issue 7

Machine Learning and Intelligent Systems
Issue 4, Issue 5, Issue 6, 2024 (Pub. June 10, 2024)

A SPECIAL ISSUE on behalf of the contribution and honor or Professor Sehie Park
Submissions starts April 02, 2024 end September 30, 2024

Recent Trends on Nonlinear Analysis and Optimization 2023
Issue 1, Issue 2

Memory of Prof. K. Goebel and Prof. W. A. Kirk, Part I; Number 12

The third anniversary of Wataru Takahashi's death, 19 NOV.

Applied Analysis and Optimizaion
Number 10

สำเนาถูกต้อง

(นายณัฐพล ปักการะโว)

Special Issue on ICNAA-2022
Number 9, 2023

Call for Papers: icFPTA

Fuzzy Systems and Data Mining (Issue 6 & 7 & 8)
Number 6, Number 7, Number 8

Nonlinear and Variational Analysis 2022 (Issue 4 & 5)
Number 4, Number 5

Issue 3, 2023

Applied Analysis and Optimization
Number 12

Nonlinear and Variational Analysis
Number 11

Machine Learning and Intelligent Systems
Number 9, 10

Mathematical contributions of Ravi P. Agarwal
Number 6, 7, 8

Recent Trends on Nonlinear Analysis and Optimization 2021
Number 4
Number 3

Memory of Prof. Wataru Takahashi
Number 2, 2022- Pub 25 FEB.
Number 1, 2022- Pub 31 DEC.
Number 12, 2021- Pub 30 NOV.
Number 11, 2021- Pub 19 NOV.

Big Data and Data Mining Strategies
Number 9, 10 2021 Pub 31 OCT.

Special Issue on Applied Analysis and Optimization, 2020
Number 8, 2021

-Memory of Prof. H.-C. Lai
Number 7, 2021

Special Issue on Noninear Analysis and Optimization, 2018
-Memory of Prof. H.-C. Lai Pub. 25 JUNE
Number 5 & Number 6, 2021

Special Issue on Fuzzy Convex Structures and Some Related Topics
Number 12, 2020

Special Issue on Recent Trends on Nonlinear Analysis and Optimization
Number 10, 2020

Special Issue on Fixed Point Theory and its Application 2019, Xinxiang, China
Number 9, 2020

Special Issue on FSDM 2019, Kitakyusyu, Japan
Number 7 & Number 8, 2020

Special Issue in Honor of the 85th Birthday of Professor Franco Giannessi
Number 5, 2020

Special Issue on IWNVA 2019, Tianjin, China
Number 4, 2020

Special Issue on IWAAO 2019
Number 2, 2020

Special Issue in Honor of the 65th Birthday of Prof. Do Sang Kim
Number 1, 2020

Special Issue in Memory of Professor Siegfried Schaible
Number 12

Special Issue on ICAA2018, Kirsehir, Turkey 2018
Number 11

Special Issue on Fixed Point Theory and its Application, Newcastle, 2017
Number 10 Pub 31 October

Special Issue in Honor of the 75th Birthday of Prof. Wataru Takahashi
Number 8 & Number 9 Pub 26 September

Special Issues on FSDM2018 (Number 5, 6, 7) Pub 31 July

Conference News

Call for Papers: icFPTA
The 11th International Conference on
NACA-ICOTA 2019
Nonlinear Analysis and Convex Analysis (NACA)
Optimization: Techniques and Applications (ICOTA)
26-31 August, 2019 at Hakodate, Japan

สำเนาถูกต้อง
(นายณัฐพล ปักการะนัง)

NEW	[Page charge(Order form) (2024)]	style file	[JNCA-sample.file]

Notice: If you need the pdf file of your article, you have to choose F-5 or S-5 in page charge form. Or please order online Journal of JNCA.

If your article is accepted please change your tex file to the style of JNCA (You can find the style in sample file and please use jnca-smple.file). If you did not use JNCA style file or we add professional edits to your tex fies, you should pay the following page charge

If you have received a proof copy of your paper from the publisher, you cannot withdraw the paper at the author's discretion. Please note the page charge form or email from the publisher for more information.

Special issue for conference.
Please note:
The publication rules for JNCA's regular issues do not apply to special issues.
For details, please contact the Guest Editor of the special issue.
Please submit papers only participants in the conference.

สำเนาถูกต้อง

(นายณัฐพล ปักการะนัง)

Volume 26, Number 10, 2025
Contents

Fixed Point Theory and Convex Optimization with Applications
On behalf of the contribution and honor for celebration
the 65th birthday of Professor Suthep Suantai

Guest Editor: A. Kaewkhao, W. Inthakon, B. Panyanak, P. Cholamjiak, W. Cholamjiak

Number 11 (continued)

สำเนาถูกต้อง

(นายณัฐพล ปักการะนัง)

Back

Yokohama Publishers

สำเนาถูกต้อง

(นายณัฐพล ปักกาพะนัง)

Online Journal

Journal of Nonlinear and Convex Analysis

Volume 26, Number 10, 2943-2957, 2025

Unified approaches to equilibrium and fixed-point problems: an improved viscosity-based subgradient extragradient framework

Nattawut Pholasa and Nuttapol Pakkaranang

ABSTRACT. This paper develops improved extragradient-type methods for solving equilibrium problems in real Hilbert spaces, where the feasible set is defined by the fixed points of a demicontractive mapping. The proposed algorithms extend the subgradient extragradient framework by incorporating inertial and viscosity terms together with a self-adaptive stepsize rule that updates automatically at each iteration. Unlike classical schemes, the methods require neither prior knowledge of Lipschitz-type constants nor any line search procedures. Strong convergence is established under mild conditions, and numerical experiments confirm the efficiency and stability of the proposed algorithms compared with existing approaches.

Download

ONLINE SUBSCRIPTION (Library Only) PDF

PDF

Open Access (Free) Html

HTML

For author

สำเนาถูกต้อง

(นายณัฐพล ปักการะนัง)

สำเนาถูกต้อง

(นายณัฐพล ปักการะนัง)

Charles Castaing
Department de Mathematiques, Case 051 Universite Montpellier II F-34095,
Montpellier Cedex 5, France
E-mail: `castaing@math.univ-montp2.fr`

Sompong Dhompongsa
Department of Mathematics, Faculty of Science, Chiang Mai University,
Chiang Mai, 50200, Thailand
E-mail: `sompongd@chiangmai.ac.th`

Masao Fukushima
Department of Applied Mathematics and Physics, Graduate School of Informatics,
Kyoto University, Kyoto 606-8501, Japan
E-mail: `fuku@amp.i.kyoto-u.ac.jp`

Norimichi Hirano
Graduate School of Environment and Information Sciences, Yokohama National University
Tokiwadai, Hodogayaku, Yokohama, Japan
E-mail: `hirano@hiranolab.jks.ynu.ac.jp`

Erdal Karapinar
Department of Mathematics, Cankaya University, Ankara, Turkey
E-mail: `erdalkarapinar@yahoo.com`

Hidefumi Kawasaki
Graduate School of Mathematics, Kyushu University, Hakozaki 6-10-1,
Fukuoka 812-8581, Japan
E-mail: `kawasaki@math.kyushu-u.ac.jp`

Do Sang Kim
Department of Applied Mathematics, Pukyong National University, Pusan, 608-737,
Republic of Korea
E-mail: `dskim@pknu.ac.kr`

Yoshikazu Kobayashi
Deapartment of Mathematics, Faculty of Science and Engineering, Chuo University,
1-13-27 Kasuga, Bunkyo-ku, Tokyo 112-8551, Japan
E-mail: `kobayashi@math.chuo-u.ac.jp`

Lai-Jiu Lin
Department of Mathematics, National Changhua University of Education, Changhua,
50058, Taiwan
E-mail: `maljlin@cc.ncue.edu.tw`

Fon-Che Liu
Institute of Mathematics, Academia Sinica Nankang, Taipei 11529, Taiwan
E-mail: `maliufc@ccvax.sinica.edu.tw`

Zuhair Nashed
Department of Mathematics, Mathematical Sciences building, University of Central Florida,
Orlando, Florida 32816-1364
E-mail address: `M.Nashed@ucf.edu`

Toshihiko Nishishiraho
Department of Mathematical Sciences, Faculty of Science, University of the Ryukyus,
Nishihara-Cho, Okinawa 903-0213, Japan

สำเนาถูกต้อง

(นายณัฐพล ปกการะนัง)

Roger Nussbaum
Department of Mathematics, Rutgers University, New Brunswick, NJ 08903-2101, U.S.A.
E-mail: nussbaum@math.rutgers.edu

Sehie Park
National Academy of Sciences, Republic of Korea and School of Mathematical Sciences, Seoul National University, Seoul 151-747, Korea
E-mail: shpark@math.snu.ac.kr

Jean-Paul Penot
Mathématiques, Faculté des Sciences, Av. de l'Université 64000 PAU, France
E-mail: jean-paul.penot@univ-pau.fr

Leon A. Petrosjan
Faculty of Applied Mathematics, & Control Processes, St. Petersburg State University, St. Petersburg, Russia
E-mail: spbuoasis7@peterlink.ru

Simeon Reich
Department of Mathematics, The Technion - Israel Institute of Technology, 32000 Haifa, Israel
E-mail: sreich@techunix.technion.ac.il

Biagio Ricceri
Department of Mathematics, Universita di Catania, Italy
E-mail: RICCERI@dipmat.unict.it

Mau-Hsiang Shih
Department of Mathematics, National Taiwan Normal University
88, Sec.4, Ting Chou Road, Taipei, Taiwan
E-mail: mhshih@math.ntnu.edu.tw

Brailey Sims
Department of Mathematics, University of Newcastle, Newcastle NSW2308, Australia
E-mail: bsims@frey.newcastle.edu.au

Gautam Srivastava
Brandon University, Brandon, Manitoba, Canada
E-mail: srivastavag@brandonu.ca

Hari Mohan Srivastava
Department of Mathematics and Statistics, University of Victoria
Victoria, British Columbia, V8W 3R4, Canada
E-mail: harimsri@math.uvic.ca

Tetsuzo Tanino
Department of Electrical, Electronic and Information Engineering, Graduate School of Engineering, Osaka University, Suita, Osaka 565-0871, Japan
E-mail: tanino@eei.eng.osaka-u.ac.jp

สำเนาถูกต้อง

(นายณัฐพล ปักการะนัง)

Michel A. Thera
Univ. de Limoges, Laboratoire d'Arithmetique, Calcul formel et Optimisation
123 avenue, Albert Thomas F-87060 LIMOGES Cedex, France
E-mail: `thera@unilim.fr`

Lionel Thibault
Départment de Mathématiques, Universit Montpellier II, CC 051, Place Eugène Bataillon, 34095 Montpellier Cedex 5, France
E-mail address: `lionel.thibault@univ-montp2.fr`

Hong-Kun Xu
Department of Mathematics, School of Science, Hangzhou Dianzi University, Hangzhou 310018 China
E-mail address: `xuhk@hdu.edu.cn`

Jen-Chih Yao
Department of Applied Mathematics, National Sun Yat-sen University, Kaohsiung 80424, Taiwan
E-mail: `yaojc@math.nsysu.edu.tw`

Coordinators

Toshi Takahashi
Yokohama Publishers, 101, 6-27 Satsukigaoka, Aoba-ku Yokohama 227-0053, Japan

Contact to: toshi@yokohamapublishers.jp

สำเนาถูกต้อง

(นายณัฐพล ปักการะนัง)

Journal of Nonlinear and Convex Analysis
Volume 26, Number 10, 2025, 2943–2957

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

UNIFIED APPROACHES TO EQUILIBRIUM AND FIXED-POINT PROBLEMS: AN IMPROVED VISCOSITY-BASED SUBGRADIENT EXTRAGRADIENT FRAMEWORK

NATTAWUT PHOLASA AND NUTTAPOL PAKKARANANG*

Abstract. This paper develops improved extragradient-type methods for solving equilibrium problems in real Hilbert spaces, where the feasible set is defined by the fixed points of a demicontractive mapping. The proposed algorithms extend the subgradient extragradient framework by incorporating inertial and viscosity terms together with a self-adaptive stepsize rule that updates automatically at each iteration. Unlike classical schemes, the methods require neither prior knowledge of Lipschitz-type constants nor any line search procedures. Strong convergence is established under mild conditions, and numerical experiments confirm the efficiency and stability of the proposed algorithms compared with existing approaches.

1. Introduction

This study investigates iterative schemes for approximating equilibrium solutions over constrained sets arising from fixed-point problems involving ρ-demicontractive mappings. Let $\mathcal{Z}$ be a real Hilbert space and $\mathcal{X} \subset \mathcal{Z}$ a nonempty, closed, and convex subset. For a bifunction $\mathcal{P} : \mathcal{Z} \times \mathcal{Z} \to \mathbb{R}$ satisfying $\mathcal{P}(s_1, s_1) = 0$ for all $s_1 \in \mathcal{X}$, the equilibrium problem (EP) associated with $\mathcal{P}$ is to find $\hat{\hbar} \in \mathcal{X}$ such that

$$\mathcal{P}(\hat{\hbar}, s_1) \geq 0, \qquad \forall s_1 \in \mathcal{X}. \tag{1.1}$$

Problem (1.1), also known as the Ky Fan inequality [9], provides a unified framework that encompasses variational inequalities, Nash equilibria, optimization, complementarity, and fixed-point problems [3, 19]. Due to its wide applicability, extensive studies have been devoted to its analysis and computation; see, e.g., [5–8, 14, 17, 21, 26, 36].

Flåm [10] and Tran et al. [32] introduced, in Euclidean spaces, an extragradient-type scheme generating a sequence $\{s_k\}$ with a fixed stepsize $0 < \varpi < \min\{\frac{1}{2c_1}, \frac{1}{2c_2}\}$:

$$\begin{cases} h_k = \arg\min\limits_{h \in \mathcal{X}} \left\{ \varpi \mathcal{P}(s_k, h) + \frac{1}{2}\|s_k - h\|^2 \right\}, \\ s_{k+1} = \arg\min\limits_{h \in \mathcal{X}} \left\{ \varpi \mathcal{P}(h_k, h) + \frac{1}{2}\|s_k - h\|^2 \right\}. \end{cases} \tag{1.2}$$

2020 *Mathematics Subject Classification.* 65Y05, 47H05, 65K15, 68W10, 47H10.

Key words and phrases. Equilibrium problem, pseudomonotone bifunction, extragradient method, Lipschitz-type condition, fixed-point problem.

This research was supported by University of Phayao and Thailand Science Research and Innovation Fund (Fundamental Fund 2025, Grant No. 5120/2567).

*Corresponding author.

(Last Pages)

สำเนาถูกต้อง

(นายณัฐพล ปักการะนัง)

This two-step structure follows the Korpelevich extragradient paradigm [15]. However, its admissible range for ϖ depends on Lipschitz-type constants that are often unknown, and the resulting sequence usually converges only weakly in Hilbert spaces.

Let $\mathcal{Q} : \mathcal{Z} \to \mathcal{Z}$ be a mapping. The associated fixed-point problem seeks $\hat{\hbar} \in \mathcal{Z}$ satisfying

$$\text{(FP)} \qquad \mathcal{Q}(\hat{\hbar}) = \hat{\hbar},$$

with solution set $\mathrm{Fix}(\mathcal{Q})$. Classical approaches include the Mann and Halpern iterations:

$$s_{k+1} = \varkappa_k s_k + (1 - \varkappa_k)\mathcal{Q}(s_k), \qquad s_{k+1} = \varkappa_k s_1 + (1 - \varkappa_k)\mathcal{Q}(s_k),$$

where $\varkappa_k \to 0$ and $\sum_{k=1}^{\infty} \varkappa_k = \infty$. The Halpern iteration ensures strong convergence in Hilbert spaces. Its modification–the viscosity method of Moudafi [18]–combines $\mathcal{Q}$ with a contraction mapping, offering a flexible framework for solving fixed-point problems in diverse applications. Other significant developments include the hybrid steepest-descent approach [38] and inertial-type methods [2, 22], which introduce momentum terms to accelerate convergence. Recent works confirm the advantages of inertial strategies in both convergence speed and stability [1,2,4,12,20,27,39,40].

Despite these advances, the method in (1.2) still faces two major drawbacks: it requires prior knowledge of the Lipschitz constant and guarantees only weak convergence. Since estimating such constants is often impractical, the method ' s applicability and efficiency are limited. These limitations naturally motivate the following question:

Can one design an inertial extragradient algorithm that achieves strong convergence using an adaptive stepsize rule for pseudomonotone equilibrium and fixed-point problems?

We answer this question affirmatively by proposing a viscosity-based subgradient extragradient framework that ensures strong convergence without relying on any Lipschitz constant. The proposed method integrates the extragradient and viscosity schemes [18], extending the results of Takahashi et al. [31] and Li et al. [16]. Our framework covers a broader class of pseudomonotone bifunctions and demicontractive mappings. By incorporating viscosity-type corrections, the scheme overcomes the weak-convergence limitation of classical Mann iterations and guarantees strong convergence. Recent developments in [13, 23–25, 28, 29, 37] further highlight the relevance of such viscosity-driven strategies for equilibrium problems.

2. Preliminaries

Let $\mathcal{X}$ be a nonempty, closed, and convex subset of a real Hilbert space $\mathcal{Z}$. Weak and strong convergence are denoted by $z_k \rightharpoonup z$ and $z_k \to z$, respectively.

Definition 2.1. Let $\mathcal{Q} : \mathcal{Z} \to \mathcal{Z}$ be an operator with a nonempty fixed-point set $\mathrm{Fix}(\mathcal{Q})$. The operator $I - \mathcal{Q}$ is said to be *demiclosed at zero* if, for any sequence $\{z_k\} \subset \mathcal{Z}$,

สำเนาถูกต้อง

$$z_k \rightharpoonup z \text{ and } (I - \mathcal{Q})z_k \to 0 \;\Rightarrow\; z \in \mathrm{Fix}(\mathcal{Q}).$$

(นายณัฐพล ปักการะนัง)

Lemma 2.2 ([30]). *Let $\{a_k\} \subset [0, \infty)$, $\{b_k\} \subset (0, 1)$, and $\{c_k\} \subset \mathbb{R}$ satisfy*

$$a_{k+1} \leq (1 - b_k)a_k + b_k c_k, \qquad \sum_{k=1}^{\infty} b_k = \infty.$$

If every subsequence $\{a_{k_j}\}$ of $\{a_k\}$ satisfies $\limsup_{j\to\infty} c_{k_j} \leq 0$ and $\liminf_{j\to\infty}(a_{k_j+1} - a_{k_j}) \geq 0$, then $\lim_{k\to\infty} a_k = 0$.

Lemma 2.3 ([33]). *Let $\daleth : \mathcal{X} \to \mathbb{R}$ be a lower semicontinuous function admitting subderivatives. A point $z_1 \in \mathcal{X}$ minimizes $\daleth$ if and only if*

$$0 \in \partial\daleth(z_1) + N_{\mathcal{X}}(z_1),$$

where $\partial\daleth(z_1)$ denotes the subdifferential of $\daleth$ at z_1 and $N_{\mathcal{X}}(z_1)$ is the normal cone to $\mathcal{X}$ at z_1.

3. Main results

We propose new inertial subgradient extragradient algorithms that incorporate both monotone and nonmonotone stepsize rules for solving

$$\text{Find } \hat{\hbar} \in \mathcal{X} \text{ such that } \mathcal{P}(\hat{\hbar}, s) \geq 0, \ \forall\, s \in \mathcal{X}, \quad \text{and} \quad \mathcal{Q}(\hat{\hbar}) = \hat{\hbar}. \tag{3.1}$$

The goal is to construct efficient iterative schemes converging strongly to a common solution of the equilibrium and fixed-point problems in (3.1). Let $\mathrm{d} : \mathcal{Z} \to \mathcal{Z}$ be a strictly contractive mapping with constant $\tau \in [0, 1)$.

To ensure convergence and existence of such a solution, we impose the following standard conditions:

($\mathcal{P}$1) Both $\mathrm{Fix}(\mathcal{Q})$ and $EP(\mathcal{X}, \mathcal{P})$ are nonempty.
($\mathcal{P}$2) $\mathcal{P}$ is *pseudomonotone*, i.e., $\mathcal{P}(z_1, z_2) \geq 0 \ \Rightarrow\ \mathcal{P}(z_2, z_1) \leq 0, \ \forall\, z_1, z_2 \in \mathcal{X}$.
($\mathcal{P}$3) $\mathcal{P}$ is *Lipschitz-type continuous* with constants $c_1, c_2 > 0$:

$$\mathcal{P}(z_1, z_3) \leq \mathcal{P}(z_1, z_2) + \mathcal{P}(z_2, z_3) + c_1\|z_1 - z_2\|^2 + c_2\|z_2 - z_3\|^2, \quad \forall\, z_1, z_2, z_3 \in \mathcal{X}.$$

($\mathcal{P}$4) $\mathcal{P}$ is weakly upper semicontinuous in the first argument, i.e., for any $\{z_k\} \subset \mathcal{X}$ with $z_k \rightharpoonup z^*$, $\limsup_{k\to\infty} \mathcal{P}(z_k, z_1) \leq \mathcal{P}(z^*, z_1), \quad \forall\, z_1 \in \mathcal{X}$.
($\mathcal{P}$5) $\mathcal{Q} : \mathcal{Z} \to \mathcal{Z}$ is *demiclosed at zero* and *ρ-demicontractive* for some $\rho \in [0, 1)$, satisfying one of the equivalent conditions:

$$\|\mathcal{Q}(z_1) - z_2\|^2 \leq \|z_1 - z_2\|^2 + \rho\,\|(I - \mathcal{Q})(z_1)\|^2, \quad \forall\, z_2 \in \mathrm{Fix}(\mathcal{Q}),$$

or equivalently, $\langle \mathcal{Q}(z_1) - z_1,\, z_1 - z_2\rangle \leq \frac{\rho - 1}{2}\,\|z_1 - \mathcal{Q}(z_1)\|^2, \quad \forall\, z_2 \in \mathrm{Fix}(\mathcal{Q})$.

สำเนาถูกต้อง

(นายณัฐพล ปกการะนัง)

Algorithm 1 Viscosity-Based Subgradient Extragradient Method

Input: Initial points $s_0, s_1 \in \mathcal{X}$; parameters $\phi \in (0,1)$, $\zeta \in (0, 2-\sqrt{2})$, $\wp \in (0,1)$, $\varpi_1 > 0$; sequence $\{\varkappa_k\}$ with $\varkappa_k \to 0$ and $\sum_{k=1}^{\infty} \varkappa_k = \infty$; and $\{\varsigma_k\} \subset (a, 1-\rho)$ for some $a > 0$.

Initialization: Set $k = 0$.

while stopping criterion not satisfied **do**

  Inertial extrapolation:

$$g_k = s_k + \phi_k(s_k - s_{k-1}), \quad 0 \le \phi_k \le \hat{\phi}_k = \begin{cases} \min\left\{\frac{\phi}{2}, \frac{\eta_k}{\|s_k - s_{k-1}\|}\right\}, & s_k \ne s_{k-1}, \\ \frac{\phi}{2}, & \text{otherwise,} \end{cases}$$

  where $\eta_k = o(\varkappa_k)$ and $\lim_{k\to\infty} \frac{\eta_k}{\varkappa_k} = 0$.

  Compute $h_k = \arg\min_{h \in \mathcal{X}} \left\{\varpi_k \mathcal{P}(g_k, h) + \frac{1}{2}\|g_k - h\|^2\right\}$.

  if $g_k = h_k$ **then**

    Stop.

  else

    Choose $\omega_k \in \partial_2 \mathcal{P}(g_k, h_k)$ s.t. $g_k - \varpi_k \omega_k - h_k \in N_{\mathcal{X}}(h_k)$, and define $\mathcal{Z}_k = \{\, z \in \mathcal{Z} : \langle g_k - \varpi_k \omega_k - h_k,\, z - h_k\rangle \le 0 \,\}$.

    Compute $r_k = \arg\min_{h \in \mathcal{Z}_k} \left\{\varpi_k \mathcal{P}(h_k, h) + \frac{1}{2}\|g_k - h\|^2\right\}$.

    Update $s_{k+1} = \varkappa_k \mathrm{d}(s_k) + (1 - \varkappa_k)\big[(1 - \varsigma_k) r_k + \varsigma_k \mathcal{Q}(r_k)\big]$.

    Adaptive stepsize:

(3.2)
$$\varpi_{k+1} = \begin{cases} \min\left\{\varpi_k, \dfrac{(2-\sqrt{2}-\zeta)\wp(\|g_k - h_k\|^2 + \|r_k - h_k\|^2)}{2[\mathcal{P}(g_k, r_k) - \mathcal{P}(g_k, h_k) - \mathcal{P}(h_k, r_k)]}\right\}, \\ \qquad\qquad\qquad\qquad \mathcal{P}(g_k, r_k) - \mathcal{P}(g_k, h_k) - \mathcal{P}(h_k, r_k) > 0, \\ \varpi_k, \qquad\qquad\qquad \text{otherwise.} \end{cases}$$

    Set $k \leftarrow k + 1$.

  end if

end while

Output: Final iterate s_k.

Lemma 3.1. *Let $\{\varpi_k\}$ be the sequence generated by* (3.2) *with limit ϖ. Then*

$$\min\left\{\frac{\wp(2-\sqrt{2}-\zeta)}{\max\{2c_1, 2c_2\}}, \varpi_1\right\} \le \varpi_k \le \varpi_1, \qquad \forall\, k \in \mathbb{N}.$$

Proof. If $\mathcal{P}(g_k, r_k) - \mathcal{P}(g_k, h_k) - \mathcal{P}(h_k, r_k) > 0$, the Lipschitz-type continuity of $\mathcal{P}$ implies $\mathcal{P}(g_k, r_k) - \mathcal{P}(g_k, h_k) - \mathcal{P}(h_k, r_k) \le c_1\|g_k - h_k\|^2 + c_2\|r_k - h_k\|^2$. Substituting into (3.2) yields

$$\varpi_{k+1} \ge \frac{\wp(2-\sqrt{2}-\zeta)}{2\max\{c_1, c_2\}}.$$

Thus $\{\varpi_k\}$ is bounded below by $\frac{\wp(2-\sqrt{2}-\zeta)}{\max\{2c_1, 2c_2\}}$ and above by ϖ_1, so it converges to some $\varpi > 0$. □

สำเนาถูกต้อง

(นายณัฐพล ปักการะนัง)

We next introduce a modified scheme, denoted as **Algorithm 2**, which replaces the monotone stepsize rule (3.2) with a nonmonotone update. Such strategies often enhance convergence speed or numerical stability. Let $\{\varphi_k\}$ be a nonnegative sequence satisfying $\sum_{k=1}^{\infty} \varphi_k < \infty$. Then the stepsize ϖ_{k+1} is updated by

(3.3)
$$\varpi_{k+1} = \begin{cases} \min\left\{ \varpi_k + \varphi_k,\ \dfrac{(2-\sqrt{2}-\zeta)\wp\big(\|g_k - h_k\|^2 + \|r_k - h_k\|^2\big)}{2\left[\mathcal{P}(g_k, r_k) - \mathcal{P}(g_k, h_k) - \mathcal{P}(h_k, r_k)\right]} \right\}, \\ \qquad\qquad\qquad\qquad \text{if } \mathcal{P}(g_k, r_k) - \mathcal{P}(g_k, h_k) - \mathcal{P}(h_k, r_k) > 0, \\ \varpi_k + \varphi_k, \qquad\qquad\quad \text{otherwise.} \end{cases}$$

Lemma 3.2. *Let $\{s_k\}$ be the sequence generated by Algorithm 1 under $(\mathcal{P}1)$–$(\mathcal{P}3)$. Then, for all $k \geq 0$,*

$$\begin{aligned} \|r_k - \hat{\hbar}\|^2 \leq{} & \|g_k - \hat{\hbar}\|^2 - \|g_k - h_k\|^2 - \|r_k - h_k\|^2 \\ & + \frac{(2-\sqrt{2}-\zeta)\varpi_k \wp(\|g_k - h_k\|^2 + \|r_k - h_k\|^2)}{\varpi_{k+1}}. \end{aligned}$$

Proof. The result follows by arguments analogous to those in Lemma 3.4 of [29]. □

Theorem 3.3. *Let $\mathcal{P} : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ be a bifunction satisfying conditions $(\mathcal{P}1)$–$(\mathcal{P}5)$. Then, the sequence $\{s_k\}$ generated by Algorithm 1 converges strongly to a point*

$$\hat{\hbar} \in \mathrm{Fix}(\mathcal{Q}) \cap \mathrm{EP}(\mathcal{X}, \mathcal{P}),$$

where $\hat{\hbar} = P_{\mathrm{EP}(\mathcal{X},\mathcal{P}) \cap \mathrm{Fix}(\mathcal{Q})}\big(\mathrm{d}(\hat{\hbar})\big)$.

Proof. It is observed that both $\mathrm{EP}(\mathcal{X}, \mathcal{P})$ and $\mathrm{Fix}(\mathcal{Q})$ are closed and convex subsets of $\mathcal{Z}$. Hence, the projection operator $P_{\mathrm{EP}(\mathcal{X},\mathcal{P}) \cap \mathrm{Fix}(\mathcal{Q})}(\mathrm{d}) : \mathcal{Z} \to \mathcal{Z}$ is a contraction. By the Banach contraction principle, there exists a unique point $\hat{\hbar} \in \mathcal{Z}$ satisfying $\hat{\hbar} = P_{\mathrm{EP}(\mathcal{X},\mathcal{P}) \cap \mathrm{Fix}(\mathcal{Q})}\big(\mathrm{d}(\hat{\hbar})\big)$. Consequently, $\hat{\hbar} \in \mathrm{EP}(\mathcal{X}, \mathcal{P}) \cap \mathrm{Fix}(\mathcal{Q})$, and it holds that

$$\langle \mathrm{d}(\hat{\hbar}) - \hat{\hbar},\, h - \hat{\hbar} \rangle \leq 0, \quad \forall\, h \in \mathrm{EP}(\mathcal{X}, \mathcal{P}) \cap \mathrm{Fix}(\mathcal{Q}).$$

Claim 1. (*Boundedness of the sequence $\{s_k\}$*) From $t_k = (1 - \varsigma_k) r_k + \varsigma_k \mathcal{Q}(r_k)$, we have

$$\begin{aligned} \|t_k - \hat{\hbar}\|^2 &= \|r_k - \hat{\hbar}\|^2 + 2\varsigma_k \langle r_k - \hat{\hbar}, \mathcal{Q}(r_k) - r_k \rangle + \varsigma_k^2 \|\mathcal{Q}(r_k) - r_k\|^2 \\ &\leq \|r_k - \hat{\hbar}\|^2 - \varsigma_k (1 - \varsigma_k - \rho) \|\mathcal{Q}(r_k) - r_k\|^2. \end{aligned} \tag{3.4}$$

By Lemma 3.2 and the convergence $\varpi_k \to \varpi$, there exists $k_1 \in \mathbb{N}$ such that

$$\begin{aligned} \|r_k - \hat{\hbar}\|^2 \leq{} & \|g_k - \hat{\hbar}\|^2 - (\sqrt{2} - 1)\big(\|g_k - h_k\|^2 + \|r_k - h_k\|^2\big) \\ & - \zeta\big(\|g_k - h_k\|^2 + \|r_k - h_k\|^2\big), \end{aligned}$$

and consequently

$$\|r_k - \hat{\hbar}\|^2 \leq \|g_k - \hat{\hbar}\|^2, \qquad k \geq k_1. \tag{3.5}$$

Combining (3.4) and (3.5) yields

$$\|t_k - \hat{\hbar}\|^2 \leq \|g_k - \hat{\hbar}\|^2 - \varsigma_k (1 - \varsigma_k - \rho) \|\mathcal{Q}(r_k) - r_k\|^2.$$

สำเนาถูกต้อง

(นายณัฐพล ปักการะนัง)

Since $\varsigma_k \in (a, 1-\rho)$,

$$\|t_k - \hat{\hbar}\| \leq \|g_k - \hat{\hbar}\|. \tag{3.6}$$

Using $\phi_k \|s_k - s_{k-1}\| \leq \eta_k$ and $\lim_{k\to\infty} \eta_k/\varkappa_k = 0$, we get

$$\lim_{k\to\infty} \frac{\phi_k}{\varkappa_k}\|s_k - s_{k-1}\| = 0.$$

Hence,

$$\|t_k - \hat{\hbar}\| \leq \|s_k - \hat{\hbar}\| + \varkappa_k K_1, \qquad K_1 > 0. \tag{3.7}$$

Since d is a contraction with constant $\tau \in [0, 1)$,

$$\begin{aligned}
\|s_{k+1} - \hat{\hbar}\| &= \|\varkappa_k \mathrm{d}(s_k) + (1-\varkappa_k)t_k - \hat{\hbar}\| \\
&\leq \varkappa_k \tau \|s_k - \hat{\hbar}\| + \varkappa_k \|\mathrm{d}(\hat{\hbar}) - \hat{\hbar}\| + (1-\varkappa_k)\|t_k - \hat{\hbar}\| \\
&\leq [1 - \varkappa_k(1-\tau)]\|s_k - \hat{\hbar}\| + \varkappa_k(1-\tau)\frac{\|\mathrm{d}(\hat{\hbar}) - \hat{\hbar}\| + K_1}{1-\tau}. \qquad (3.8)
\end{aligned}$$

Thus,

$$\|s_{k+1} - \hat{\hbar}\| \leq \max\left\{\|s_k - \hat{\hbar}\|, \frac{\|\mathrm{d}(\hat{\hbar}) - \hat{\hbar}\| + K_1}{1-\tau}\right\},$$

which implies that the sequence $\{s_k\}$ is bounded.

Claim 2. (*Existence of a recursive inequality*) There exists a constant $K_2 > 0$ such that

$$\begin{aligned}
&(1-\varkappa_k)(\sqrt{2}-1)\|g_k - h_k\|^2 + (1-\varkappa_k)(\sqrt{2}-1)\|r_k - h_k\|^2 \\
&\quad + (1-\varkappa_k)\varsigma_k(1-\rho-\varsigma_k)\|\mathcal{Q}(r_k) - r_k\|^2 \\
&\leq \|s_k - \hat{\hbar}\|^2 - \|s_{k+1} - \hat{\hbar}\|^2 + \varkappa_k\|\mathrm{d}(s_k) - \hat{\hbar}\|^2 + \varkappa_k K_2.
\end{aligned}$$

Indeed, from (3.7),

$$\|g_k - \hat{\hbar}\|^2 \leq (\|s_k - \hat{\hbar}\| + \varkappa_k K_1)^2 \leq \|s_k - \hat{\hbar}\|^2 + \varkappa_k K_2, \tag{3.9}$$

for some $K_2 > 0$. Combining (3.9) with (3.5) yields

$$\begin{aligned}
\|t_k - \hat{\hbar}\|^2 &\leq \|s_k - \hat{\hbar}\|^2 + \varkappa_k K_2 - (\sqrt{2}-1)(\|g_k - h_k\|^2 + \|r_k - h_k\|^2) \\
&\quad - \varsigma_k(1-\varsigma_k-\rho)\|\mathcal{Q}(r_k) - r_k\|^2. \qquad (3.10)
\end{aligned}$$

Using $s_{k+1} = \varkappa_k \mathrm{d}(s_k) + (1-\varkappa_k)t_k$, we have

$$\begin{aligned}
\|s_{k+1} - \hat{\hbar}\|^2 &\leq \varkappa_k\|\mathrm{d}(s_k) - \hat{\hbar}\|^2 + (1-\varkappa_k)\|t_k - \hat{\hbar}\|^2 \\
&\leq \varkappa_k\|\mathrm{d}(s_k) - \hat{\hbar}\|^2 + \|s_k - \hat{\hbar}\|^2 + \varkappa_k K_2 \\
&\quad - (1-\varkappa_k)\varsigma_k(1-\rho-\varsigma_k)\|\mathcal{Q}(r_k) - r_k\|^2 \\
&\quad - (1-\varkappa_k)(\sqrt{2}-1)(\|g_k - h_k\|^2 + \|r_k - h_k\|^2),
\end{aligned}$$

which, after rearrangement, gives the desired inequality.

Claim 3. (*Recursive inequality for $\{s_k\}$*) There exists a constant $K > 0$ such that

$$\|s_{k+1} - \hat{\hbar}\|^2 \leq (1 - \varkappa_k - \tau\varkappa_k)\|s_k - \hat{\hbar}\|^2$$

สำเนาถูกต้อง

(นายณัฐพล ปักการะนัง)

$$+ (1-\tau)\varkappa_k\left[\frac{3K\,\phi_k}{(1-\tau)\varkappa_k}\|s_k - s_{k-1}\| + \frac{2}{1-\tau}\big\langle \mathrm{d}(\hat{\hbar}) - \hat{\hbar},\, s_{k+1} - \hat{\hbar}\big\rangle\right].$$

From $g_k = s_k + \phi_k(s_k - s_{k-1})$, we obtain

$$\begin{aligned}\|g_k - \hat{\hbar}\|^2 &= \|s_k - \hat{\hbar}\|^2 + 2\phi_k\langle s_k - \hat{\hbar}, s_k - s_{k-1}\rangle + \phi_k^2\|s_k - s_{k-1}\|^2\\ &\le \|s_k - \hat{\hbar}\|^2 + 3K\,\phi_k\|s_k - s_{k-1}\|,\end{aligned}\tag{3.11}$$

where $K = \sup_{k\in\mathbb{N}}\{\|s_k - \hat{\hbar}\|,\ \phi_k\|s_k - s_{k-1}\|\}$.

Using (3.11) and the contractiveness of d with constant $\tau \in [0,1)$, we have

$$\begin{aligned}\|s_{k+1} - \hat{\hbar}\|^2 &= \big\|\varkappa_k \mathrm{d}(s_k) + (1-\varkappa_k)t_k - \hat{\hbar}\big\|^2\\ &\le \varkappa_k\|\mathrm{d}(s_k) - \mathrm{d}(\hat{\hbar})\|^2 + (1-\varkappa_k)\|t_k - \hat{\hbar}\|^2 + 2\varkappa_k\big\langle \mathrm{d}(\hat{\hbar}) - \hat{\hbar},\, s_{k+1} - \hat{\hbar}\big\rangle\\ &\le \varkappa_k\tau\|s_k - \hat{\hbar}\|^2 + (1-\varkappa_k)\|s_k - \hat{\hbar}\|^2 + 3K\,\phi_k\|s_k - s_{k-1}\|\\ &\quad + 2\varkappa_k\big\langle \mathrm{d}(\hat{\hbar}) - \hat{\hbar},\, s_{k+1} - \hat{\hbar}\big\rangle,\end{aligned}$$

which, after rearrangement, yields the desired inequality.

Claim 4. (*Strong convergence of* $\{s_k\}$) The sequence $\{\|s_k - \hat{\hbar}\|^2\}$ converges to zero as $k \to \infty$. Set

$$p_k := \|s_k - \hat{\hbar}\|^2, \qquad c_k := \frac{3K\phi_k}{(1-\tau)\varkappa_k}\|s_k - s_{k-1}\| + \frac{2}{1-\tau}\langle \mathrm{d}(\hat{\hbar}) - \hat{\hbar},\, s_{k+1} - \hat{\hbar}\rangle.$$

From **Claim 3**, we can rewrite $p_{k+1} \le (1-\delta_k)p_k + \delta_k c_k,\ \delta_k = (1-\tau)\varkappa_k$. By Lemma 2.2, it suffices to verify that

$$\limsup_{j\to\infty} c_{k_j} \le 0 \quad \text{for any subsequence } \{p_{k_j}\} \text{ satisfying } \liminf_{j\to\infty}(p_{k_j+1} - p_{k_j}) \ge 0.$$

Equivalently, we need to show $\limsup_{j\to\infty}\langle \mathrm{d}(\hat{\hbar}) - \hat{\hbar},\, s_{k_j+1} - \hat{\hbar}\rangle \le 0$ for every such subsequence $\{\|s_{k_j} - \hat{\hbar}\|\}$. Since $\liminf_{j\to\infty}\big(\|s_{k_j+1} - \hat{\hbar}\|^2 - \|s_{k_j} - \hat{\hbar}\|^2\big) \ge 0$, **Claim 2** yields

$$\lim_{j\to\infty}\|g_{k_j} - h_{k_j}\| = \lim_{j\to\infty}\|r_{k_j} - h_{k_j}\| = \lim_{j\to\infty}\|\mathcal{Q}(r_{k_j}) - r_{k_j}\| = 0.$$

Consequently $\|r_{k_j} - g_{k_j}\| \to 0$. From $g_k = s_k + \phi_k(s_k - s_{k-1})$ and $\phi_k\|s_k - s_{k-1}\| = o(\varkappa_k)$, we also have

$$\|g_{k_j} - s_{k_j}\| \to 0, \qquad \Rightarrow \qquad \|r_{k_j} - s_{k_j}\| \to 0.$$

Since $t_{k_j} = (1-\varsigma_{k_j})r_{k_j} + \varsigma_{k_j}\mathcal{Q}(r_{k_j})$,

$$\|t_{k_j} - r_{k_j}\| \le (1-\rho)\|\mathcal{Q}(r_{k_j}) - r_{k_j}\| \to 0.$$

Hence $\|s_{k_j+1} - s_{k_j}\| \to 0$ and consequently $\|g_{k_j} - s_{k_j+1}\| \to 0$.

From Algorithm 1,

$$\varpi_{k_j}\mathcal{P}(h_{k_j}, h) \ge \varpi_{k_j}\mathcal{P}(h_{k_j}, r_{k_j}) + \langle g_{k_j} - r_{k_j},\, h - r_{k_j}\rangle.\tag{3.12}$$

From (3.2), it follows that

$$\begin{aligned}\varpi_{k_j}\mathcal{P}(h_{k_j}, r_{k_j}) &\ge \varpi_{k_j}\mathcal{P}(g_{k_j}, r_{k_j}) - \varpi_{k_j}\mathcal{P}(g_{k_j}, h_{k_j})\\ &\qquad - \frac{\varpi_{k_j}\wp(\|g_{k_j} - h_{k_j}\|^2 + \|r_{k_j} - h_{k_j}\|^2)}{2\varpi_{k_j+1}}.\end{aligned}\tag{3.13}$$

สำเนาถูกต้อง

(นายณัฐพล ปักการะนัง)

Combining (3.12), (3.13), we obtain

$$\varpi_{k_j}\mathcal{P}(h_{k_j}, h) \geq \langle g_{k_j} - h_{k_j},\, r_{k_j} - h_{k_j}\rangle - \frac{\wp\varpi_{k_j}}{2\varpi_{k_j+1}}\|g_{k_j} - h_{k_j}\|^2 - \frac{\wp\varpi_{k_j}}{2\varpi_{k_j+1}}\|h_{k_j} - r_{k_j}\|^2 + \langle g_{k_j} - r_{k_j},\, h - r_{k_j}\rangle.$$

All terms on the right tend to zero, giving

$$0 \leq \limsup_{j\to\infty} \mathcal{P}(h_{k_j}, h) \leq \mathcal{P}(\hat{s}, h), \qquad \forall\, h \in \mathcal{Z}_k.$$

Since $\mathcal{X} \subset \mathcal{Z}_k$ and $\mathcal{P}(\hat{s}, h) \geq 0$ for all $h \in \mathcal{X}$, we have $\hat{s} \in EP(\mathcal{P}, \mathcal{X})$. Thus

$$\limsup_{j\to\infty}\langle \mathrm{d}(\hat{\hbar}) - \hat{\hbar},\, s_{k_j} - \hat{\hbar}\rangle = \langle \mathrm{d}(\hat{\hbar}) - \hat{\hbar},\, \hat{s} - \hat{\hbar}\rangle \leq 0,$$

and, since $\|s_{k_j+1} - s_{k_j}\| \to 0$,

$$\limsup_{j\to\infty}\langle \mathrm{d}(\hat{\hbar}) - \hat{\hbar},\, s_{k_j+1} - \hat{\hbar}\rangle \leq 0.$$

Hence, by Lemma 2.2 and **Claim 3**, we conclude that $s_k \to \hat{\hbar}$ strongly as $k \to \infty$. This completes the proof of Theorem 3.3. □

4. Numerical illustrations

In this section, we compare the proposed algorithms with several existing methods from the literature and examine the influence of different control-parameter settings on their computational performance. All numerical experiments were conducted in MATLAB R2018b on an HP Core$^{\mathrm{TM}}$ i5-6200U laptop with 8 GB RAM.

Example 4.1. The original problem considered here is derived from the Nash-Cournot oligopolistic equilibrium framework introduced by Tran et al. [32]. Let $w : \mathcal{Z} \to \mathbb{R}$ be a real-valued function defined by $\mathrm{lev}_{\leq w} := \{\, s \in \mathcal{Z} : w(s) \leq 0 \,\} \neq \emptyset$. We define the following piecewise subgradient projection mapping:

$$\mathcal{Q}(s) = \begin{cases} s - \dfrac{w(s)}{\|r(s)\|^2}\, r(s), & \text{if } w(s) \geq 0, \\ s, & \text{otherwise,} \end{cases}$$

where $r(s) \in \partial w(s)$. Furthermore, define the bifunction $\mathcal{F}$ by $\mathcal{F}(s,h) = \langle Ps + Qh + c,\, h - s\rangle$, where $c \in \mathbb{R}^M$, and P and Q are real matrices of order M. The iterative process is initialized with $s_0 = s_1 = (2, 2, \ldots, 2)$, and the stopping criterion is given by $D_k = \|g_k - h_k\| \leq 10^{-3}$.

Numerical experiments on Example 4.1 are conducted to evaluate the performance of Algorithms 1 and 2 in terms of execution time (denoted by t, measured in seconds) and the number of iterations (denoted by k) required for convergence. These experiments specifically aim to examine the influence of the following factors: (i) the value of ϖ_1, (ii) the value of ζ, and (ii) a comparative analysis of Algorithms 1 and 2 with other existing methods. The results provide insights into how variations in these parameters affect the efficiency and convergence characteristics of the algorithms.

สำเนาถูกต้อง

(นายณัฐพล ปกการะนัง)

For the first two experiments, the parameters of the bifunction $\mathcal{F}$ are specified by the following matrices and vector:

$$P = \begin{pmatrix} 3.1 & 2 & 0 & 0 & 0 \\ 2 & 3.6 & 0 & 0 & 0 \\ 0 & 0 & 3.5 & 2 & 0 \\ 0 & 0 & 2 & 3.3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}, \qquad Q = \begin{pmatrix} 1.6 & 1 & 0 & 0 & 0 \\ 1 & 1.6 & 0 & 0 & 0 \\ 0 & 0 & 1.5 & 1 & 0 \\ 0 & 0 & 1 & 1.5 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}, \qquad c = \begin{pmatrix} 1 \\ -2 \\ -1 \\ 2 \\ -1 \end{pmatrix}.$$

The feasible set $\mathcal{X} \subset \mathbb{R}^M$ is defined as $\mathcal{X} := \{ s \in \mathbb{R}^M : -2 \leq s_i \leq 5 \}$.

$\boxed{\textit{Experiment 1.}}$ *(Variation of the parameter* $\boldsymbol{\varpi_1}$*)* In this experiment, we examine the influence of the initial stepsize parameter ϖ_1 on the computational performance of the proposed algorithms. The following parameter values are used throughout the experiment: $\phi = 0.45$, $\wp = 0.465$, $\zeta = 0.237$, $\eta_k = \frac{5}{(1+k)^2}$, $\varsigma_k = \frac{1-\rho}{2}$, $\varkappa_k = \frac{1}{2k+3}$, $\mathrm{d}(s) = \frac{s}{3}$, $\varphi_k = \frac{100}{(1+k)^2}$. The numerical tests are carried out for ten distinct initial stepsize values of ϖ_1. The results are illustrated in Figure 1 and summarized in Table 1. The analysis of these results leads to the following observations: (i) The computational efficiency, measured in terms of iteration count, is highly sensitive to the selection of ϖ_1. In particular, the number of iterations required by Algorithm 1 decreases noticeably as ϖ_1 approaches zero. (ii) Similarly, the total CPU execution time exhibits a decreasing trend with smaller values of ϖ_1, indicating that Algorithm 1 converges faster for smaller initial stepsizes. (iii) Algorithm 2 also demonstrates improved efficiency in both iteration count and execution time when ϖ_1 is reduced, confirming a consistent dependency of convergence performance on this parameter. This experiment highlights the critical influence of ϖ_1 on the convergence rate and overall computational cost of the proposed algorithms. Hence, an appropriate selection or adaptive tuning of ϖ_1 can significantly enhance the stability and efficiency of the iterative process.

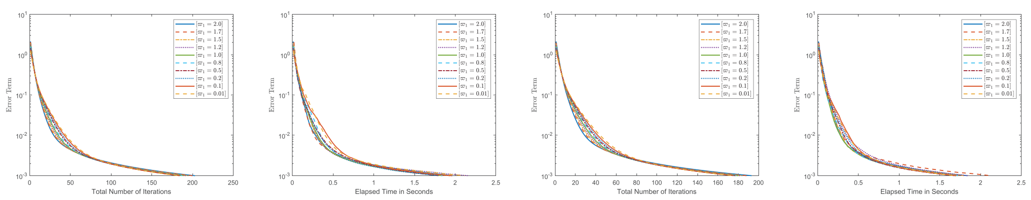


FIGURE 1. Performance of Algorithm 1 and Algorithm 2 with varying initial stepsize ϖ_1. As ϖ_1 decreases from 2.0 to 0.01, the plots show fewer iterations and shorter CPU times, indicating that smaller stepsizes enhance convergence speed and reduce computational cost.

$\boxed{\textit{Experiment 2.}}$ *(Variation of the parameter* $\boldsymbol{\zeta}$*)* In this experiment, we investigate the effect of the control parameter ζ on the convergence behavior and computational efficiency of the proposed algorithms. The following parameter settings are used throughout the experiment: $\varpi_1 = 0.55$, $\phi = 0.45$, $\wp = 0.465$, $\eta_k =$

สำเนาถูกต้อง

(นายณัฐพล ปักการะนัง)

TABLE 1. Iteration count (k) and time (t) (sec) for `OurAlg1` and `OurAlg2` under varying ϖ_1.

ϖ_1	2.0	1.7	1.5	1.2	1.0	0.8	0.5	0.2	0.1	0.01
`OurAlg1` (k)	203	200	195	192	189	187	185	184	184	183
`OurAlg1` (t)	1.96	1.94	1.93	2.16	1.84	1.85	1.79	1.79	1.96	2.11
`OurAlg2` (k)	193	188	184	182	180	178	177	176	175	175
`OurAlg2` (t)	1.84	2.13	1.80	1.85	1.70	1.72	1.68	1.69	1.70	1.69

$\frac{5}{(1+k)^2}$, $\varsigma_k = \frac{1-\rho}{2}$, $\varkappa_k = \frac{1}{2k+3}$, $\mathrm{d}(s) = \frac{s}{3}$, $\varphi_k = \frac{100}{(1+k)^2}$. The numerical outcomes corresponding to ten distinct values of ζ are displayed in Figure 2 and summarized in Table 2. Based on these results, the following observations can be made: (i) The computational efficiency, measured in terms of iteration count, is influenced by the choice of ζ. As ζ decreases towards zero, the number of iterations required for Algorithm 1 gradually increases, indicating slower convergence. (ii) The total CPU execution time exhibits a similar dependency on ζ, with higher computational costs observed for smaller ζ values. This suggests that the damping effect associated with ζ plays a stabilizing role in accelerating convergence. (iii) Algorithm 2 also shows a consistent trend: both iteration count and execution time increase as ζ approaches smaller values, implying that appropriate tuning of ζ is crucial for achieving efficient performance.

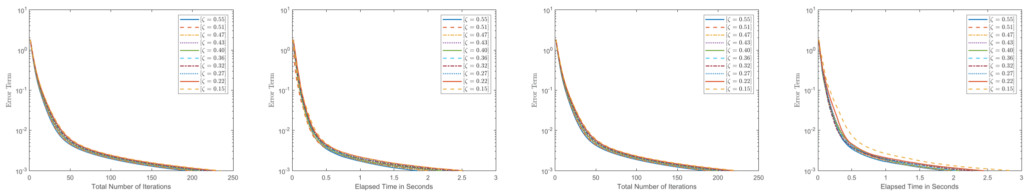


FIGURE 2. Comparison of Algorithm 1 and Algorithm 2 under varying ζ values. As ζ decreases from 0.55 to 0.15, iteration counts and CPU times increase, showing that smaller ζ slows convergence and raises computational cost.

TABLE 2. Iteration count (k) and time (t) (sec) for `OurAlg1` and `OurAlg2` under varying ζ values.

ζ	0.55	0.51	0.47	0.43	0.40	0.36	0.32	0.27	0.22	0.15
`OurAlg1` (k)	192	196	200	205	209	214	219	224	229	235
`OurAlg1` (t)	1.84	2.04	2.00	2.12	2.20	2.37	2.39	2.52	2.52	2.53
`OurAlg2` (k)	184	187	191	195	200	204	209	213	219	224
`OurAlg2` (t)	1.91	1.93	1.94	2.00	2.08	2.06	2.28	2.40	2.44	2.86

$\boxed{\textit{Experiment 3.}}$ *(Numerical comparison of Algorithms 1 and 2).* In this final experiment, we perform a comparative numerical evaluation between the proposed algorithms and several well-established methods from the literature. The objective is to assess the relative computational performance of each method in terms of both iteration count and CPU execution time across different dimensions of

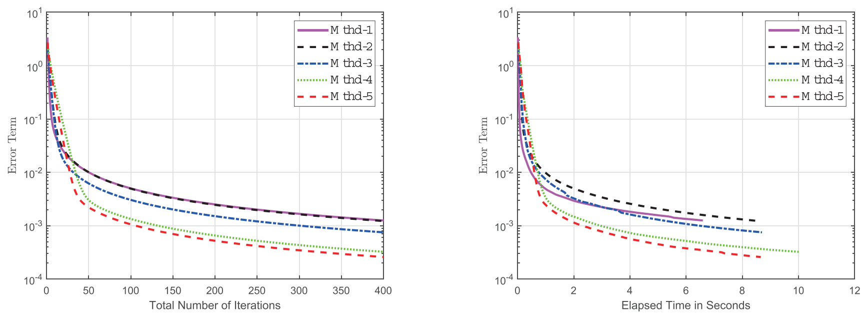
สำเนาถูกต้อง

(นายณัฐพล ปักการะนัง)

the real Hilbert space. The parameters are defined as follows: (i) Mthd-1 (Algorithm 1 in [35]): $\varkappa_k = \frac{1}{3k+5}, \varsigma_k = \frac{1}{4}, \varpi_k = \min\left\{\frac{1}{4c_1}, \frac{1}{4c_2}\right\}$; (ii) Mthd-2 (Algorithm 2 in [34]): $\varkappa_k = \frac{1}{3k+5}, \varsigma_k = \frac{1}{4}, \varpi_k = \min\left\{\frac{1}{4c_1}, \frac{1}{4c_2}\right\}$; (iii) Mthd-3 (Algorithm 2 in [11]): $\varkappa_k = \frac{1}{3k+5}, \varsigma_k = \frac{1}{4}, \varpi_k = \min\left\{\frac{1}{4c_1}, \frac{1}{4c_2}\right\}$; (iv) Mthd-4 (Algorithm 1): $\varpi_1 = 0.55, \phi = 0.65, \wp = 0.565, \zeta = 0.357, \eta_k = \frac{10}{(2+k)^2}, \varsigma_k = \frac{1-\rho}{2}, \varkappa_k = \frac{1}{3k+5}, \mathrm{d}(s) = \frac{s}{4}$; (v) Mthd-5 (Algorithm 2): $\varpi_1 = 0.55, \phi = 0.65, \wp = 0.565, \zeta = 0.357, \eta_k = \frac{10}{(2+k)^2}, \varsigma_k = \frac{1-\rho}{2}, \varkappa_k = \frac{1}{3k+5}, \mathrm{d}(s) = \frac{s}{4}, \varphi_k = \frac{100}{(1+k)^2}$.
The numerical tests were conducted for varying dimensions of the Hilbert space, specifically M = 5, 10, 30, 50, and 100. The results are presented in Figures 3 and summarized in Table 3. A detailed examination of the outcomes reveals the following key insights: (i) Algorithm 1 (Mthd-4) consistently achieves better performance compared to classical methods (Mthd-1–Mthd-3), requiring fewer iterations and shorter execution times across all tested dimensions. (ii) Algorithm 2 (Mthd-5) demonstrates the best overall performance among all tested methods. In most cases, it requires the least number of iterations and achieves the lowest CPU time, confirming its superior computational efficiency and stability.

TABLE 3. Iteration count (k) and time (t) (sec) for five methods across different space dimensions M. Algorithms Mthd-4 (1) and Mthd-5 (Algorithm 2) achieve faster convergence, particularly in higher dimensions.

M	Mthd-1		Mthd-2		Mthd-3		Mthd-4		Mthd-5	
	(k)	(t)	(k)	(t)	(k)	(t)	(k)	(t)	(k)	(t)
5	400	6.593	400	8.592	400	8.699	400	10.006	400	8.655
10	500	8.883	500	11.096	500	11.199	500	11.150	500	11.206
30	600	30.999	600	17.883	600	17.745	600	18.682	600	13.159
50	1500	34.418	1500	37.661	1500	37.599	1500	47.401	1500	47.118
100	2000	46.374	2000	80.306	2000	51.765	2000	53.010	2000	51.988



(A) Iteration counts for $M = 5$. (B) Execution times (seconds) for $M = 5$.

FIGURE 3. Comparison of five methods for $M = 5$ in Example 4.1. All show similar iteration counts, but the proposed Mthd-4 and Mthd-5 converge faster with lower computational cost.

สำเนาถูกต้อง

(นายณัฐพล ปักการะนัง)

Example 4.2. Let $\mathcal{Z} = L^2([0,1])$ denote a real Hilbert space with inner product $\langle a, b\rangle = \int_0^1 a(t)\, b(t)\, dt$ for all $a, b \in \mathcal{Z}$, and induced norm $\|a\| = \left(\int_0^1 |a(t)|^2\, dt\right)^{1/2}$. Consider the mapping $\mathcal{S} : \mathcal{X} \to \mathcal{Z}$ defined by $\mathcal{S}(a)(t) = \int_0^1 \big(a(t) - H(t,a)\, f(a(t))\big)\, da + g(t)$, where $\mathcal{X} := \{\, a \in L^2([0,1]) : \|a\| \leq 1 \,\}$ and

$$H(t,a) = \frac{2ta\, e^{t+a}}{e\sqrt{e^2-1}}, \qquad f(a) = \cos a, \qquad g(t) = \frac{2t\, e^{t}}{e\sqrt{e^2-1}}.$$

Define the bifunction $\mathcal{F}(a,h) := \big\langle \mathcal{S}(a),\, h - a\big\rangle$ for all $a, h \in \mathcal{X}$. Let $\mathcal{Q} : L^2([0,1]) \to L^2([0,1])$ be given by $\mathcal{Q}(a)(t) = \int_0^1 t\, a(u)\, du, \quad t \in [0,1]$.

Figure 4 and Table 4 report numerical results for three different initial functions: $s_0 = t^2$, $s_0 = t\sin(t)$, and $s_0 = t\cos(t)$. A detailed inspection leads to the following observations: (i) Algorithm 1 outperforms existing methods in terms of both iteration count and CPU time across the tested initializations. (ii) Algorithm 2 further improves upon Algorithm 1, typically achieving fewer iterations and notably shorter run times. These results highlight the effectiveness of the proposed algorithms–particularly Algorithm 2–in improving computational efficiency for infinite-dimensional problems.

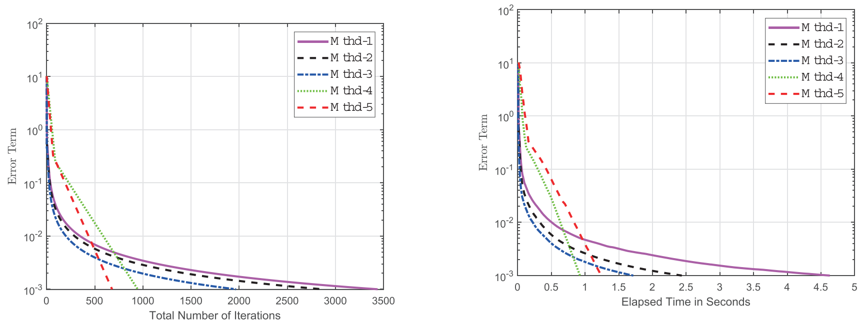
Parameter settings. The following configurations are used in this experiment: (i) Mthd-1 (Algorithm 1 in [35]): $\varkappa_k = \frac{1}{5k+7}, \varsigma_k = \frac{1}{2}, \varpi_k = \min\left\{\frac{1}{4c_1}, \frac{1}{4c_2}\right\}$. (ii) Mthd-2 (Algorithm 2 in [34]): $\varkappa_k = \frac{1}{5k+7}, \varsigma_k = \frac{1}{2}, \varpi_k = \min\left\{\frac{1}{4c_1}, \frac{1}{4c_2}\right\}$. (iii) Mthd-3 (Algorithm 2 in [11]): $\varkappa_k = \frac{1}{5k+7}, \varsigma_k = \frac{1}{2}, \varpi_k = \min\left\{\frac{1}{4c_1}, \frac{1}{4c_2}\right\}$. (iv) Mthd-4 (Algorithm 1): $\varpi_1 = 0.35, \phi = 0.75, \wp = 0.565, \zeta = 0.457, \eta_k = \frac{100}{(2+k)^2}, \varsigma_k = \frac{1-\rho}{2}, \varkappa_k = \frac{1}{5k+7}, \mathrm{d}(s) = \frac{s}{2}$. (v) Mthd-5 (Algorithm 2): $\varpi_1 = 0.35, \phi = 0.75, \wp = 0.565, \zeta = 0.457, \eta_k = \frac{100}{(2+k)^2}, \varsigma_k = \frac{1-\rho}{2}, \varkappa_k = \frac{1}{5k+7}, \mathrm{d}(s) = \frac{s}{2}, \varphi_k = \frac{200}{(1+k)^2}$.

Table 4. Performance summary for Example 4.2. Iteration count (k) and CPU time (t) (sec) are shown for $s_0 = s_1 \in t^2, , t\sin(t), , t\cos(t)$. Among baseline methods (Mthd-1–Mthd-3), Mthd-3 performs best, while the proposed Mthd-4 (Algorithm 1) achieves the lowest k and t for $s_0 = t^2$ and remains competitive for oscillatory cases.

$s_0 = s_1$	Mthd-1		Mthd-2		Mthd-3		Mthd-4	
	(k)	(t)	(k)	(t)	(k)	(t)	(k)	(t)
t^2	3438	4.631	2886	2.435	1967	1.716	946	0.924
$t\sin(t)$	2000	2.838	2000	1.695	2000	1.786	2000	1.924
$t\cos(t)$	1500	2.071	1500	1.633	1500	1.391	1500	1.394

สำเนาถูกต้อง

(นายณัฐพล ปักการะนัง)



(A) Iteration counts for $s_0 = s_1 = t^2$.

(B) Execution times (seconds) for $s_0 = s_1 = t^2$.

FIGURE 4. Comparison of all methods in Example 4.2 with polynomial initialization $s_0 = t^2$. The proposed Mthd-4 (Algorithm 1) achieves markedly fewer iterations and lower CPU time, demonstrating faster convergence for smooth initial data.

Conclusion

This study introduced a new class of explicit extragradient-type algorithms for simultaneously solving an equilibrium problem defined by a pseudomonotone and Lipschitz-type bifunction, together with a fixed-point problem involving a ρ-demicontractive mapping in a real Hilbert space. A novel stepsize selection strategy was proposed, which removes the need for prior knowledge of the Lipschitz-type constant. Under appropriate assumptions, strong convergence theorems were established for the generated sequences, guaranteeing convergence to a common solution of the studied problems. Comprehensive numerical experiments demonstrated the superior performance of the proposed algorithms compared with existing approaches. In particular, the use of a non-monotone adaptive stepsize significantly enhanced convergence speed and computational stability. Overall, the proposed framework provides an efficient and practical approach for addressing challenging equilibrium and fixed-point problems in real Hilbert spaces.

References

[1] F. Alvarez, *An inertial proximal method for maximal monotone operators via discretization of a nonlinear oscillator with damping*, Set-Valued Var. Anal. **9** (2001), 3–11.

[2] A. Beck and M. Teboulle, *A fast iterative shrinkage-thresholding algorithm for linear inverse problems*, SIAM J. Imaging Sci. **2** (2009), 183–202.

[3] E. Blum, *From optimization and variational inequalities to equilibrium problems*, Math. Student **63** (1994), 123–145.

[4] L. C. Ceng and J.-C. Yao, *A hybrid iterative scheme for mixed equilibrium problems and fixed point problems*, J. Comput. Appl. Math. **214** (2008), 186–201.

[5] Q. L. Dong, Y. J. Cho, L. L. Zhong and T. M. Rassias, *Inertial projection and contraction algorithms for variational inequalities*, J. Global Optim. **70** (2017), 687–704.

[6] Q. L. Dong, J. Z. Huang, X. H. Li, Y. J. Cho and T. M. Rassias, *MiKM: Multi-step inertial Krasnosel'skiĭ–Mann algorithm and its applications*, J. Global Optim. **73** (2018), 801–824.

[7] F. Giannessi, A. Maugeri and P. M. Pardalos, *Equilibrium Problems: Nonsmooth Optimization and Variational Inequality Models*, Springer Science & Business Media, New York, 2006.

สำเนาถูกต้อง

(นายณัฐพล ปักการะนัง)

[8] J. Fan, L. Liu and X. Qin, *A subgradient extragradient algorithm with inertial effects for solving strongly pseudomonotone variational inequalities*, Optimization **68** (2019), 1–17.

[9] K. Fan, *A minimax inequality and applications*, in: *Inequalities III*, O. Shisha (ed.), Academic Press, New York, 1972.

[10] S. D. Flåm and A. S. Antipin, *Equilibrium programming using proximal-like algorithms*, Math. Program. **78** (1996), 29–41.

[11] D. V. Hieu, *Strong convergence of a new hybrid algorithm for fixed point problems and equilibrium problems*, Math. Model. Anal. **24** (2018), 1–19.

[12] D. V. Hieu, Y. J. Cho and Y. B. Xiao, *Modified extragradient algorithms for solving equilibrium problems*, Optimization **67** (2018), 2003–2029.

[13] C. Izuchukwu, G. N. Ogwo and B. Zinsou, *A new inertial condition on the subgradient extragradient method for solving pseudomonotone equilibrium problem*, Commun. Nonlinear Sci. Numer. Simul. **135** (2024): 108076.

[14] I. V. Konnov, *Application of the proximal point method to nonmonotone equilibrium problems*, J. Optim. Theory Appl. **119** (2003), 317–333.

[15] G. M. Korpelevich, *The extragradient method for finding saddle points and other problems*, Ekon. Mat. Metody **12** (1976), 747–756.

[16] M. Li and Y. Yao, *Strong convergence of an iterative algorithm for λ-strictly pseudo-contractive mappings in Hilbert spaces*, An. St. Univ. Ovidius Constanţa **18** (2010), 219–228.

[17] A. Moudafi, *Proximal point algorithm extended to equilibrium problems*, J. Nat. Geom. **15** (1999), 91–100.

[18] A. Moudafi, *Viscosity approximation methods for fixed-point problems*, J. Math. Anal. Appl. **241** (2000), 46–55.

[19] L. D. Muu and W. Oettli, *Convergence of an adaptive penalty scheme for finding constrained equilibria*, Nonlinear Anal. **18** (1992), 1159–1166.

[20] N. Pakkaranang, *Double inertial extragradient algorithms for solving variational inequality problems with convergence analysis*, Math. Methods Appl. Sci. **47** (2024), 11642–11669.

[21] B. Panyanak, C. Khunpanuk, N. Pholasa and N. Pakkaranang, *Dynamical inertial extragradient techniques for solving equilibrium and fixed-point problems in real Hilbert spaces*, J. Inequal. Appl. **2023** (2023): Article ID 7.

[22] B. T. Polyak, *Some methods of speeding up the convergence of iteration methods*, USSR Comput. Math. Math. Phys. **4** (1964), 1–17.

[23] H. Rehman, D. Ghosh, J. C. Yao and X. Zhao, *Extragradient methods with dual inertial steps for solving equilibrium problems*, Appl. Anal. **103** (2024), 2941–2976.

[24] H. Rehman, D. Ghosh, J. C. Yao and X. Zhao, *Solving equilibrium and fixed-point problems in Hilbert spaces: A class of strongly convergent Mann-type dual-inertial subgradient extragradient methods*, J. Comput. Appl. Math. **464** (2025): 116509.

[25] H. Rehman, P. Kumam, V. Berinde and W. Kumam, *Non-monotonic and self-adaptive strongly convergent iterative methods for efficiently solving variational inequalities with pseudomonotone operators*, Comput. Appl. Math. **43** (2024): Article ID 60.

[26] H. Rehman, P. Kumam, Y. J. Cho, Y. I. Suleiman and W. Kumam, *Modified Popov's explicit iterative algorithms for solving pseudomonotone equilibrium problems*, Optim. Methods Softw. **36** (2020), 82–113.

[27] H. Rehman, P. Kumam, Y. J. Cho and P. Yordsorn, *Weak convergence of explicit extragradient algorithms for solving equilibrium problems*, J. Inequal. Appl. **2019** (2019): Article ID 1.

[28] H. Rehman, P. Kumam, Y. Shehu, M. Ozdemir and W. Kumam, *An inertial non-monotonic self-adaptive iterative algorithm for solving equilibrium problems*, J. Nonlinear Var. Anal. **6** (2022), 51–67.

[29] H. Rehman, B. Tan and J. C. Yao, *Relaxed inertial subgradient extragradient methods for equilibrium problems in Hilbert spaces and their applications to image restoration*, Commun. Nonlinear Sci. Numer. Simul. **146** (2025), 108795.

[30] S. Saejung and P. Yotkaew, *Approximation of zeros of inverse strongly monotone operators in Banach spaces*, Nonlinear Anal. **75** (2012), 742–750.

สำเนาถูกต้อง

(นายณัฐพล ปักการะนัง)

[31] S. Takahashi and W. Takahashi, *Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces*, J. Math. Anal. Appl. **331** (2007), 506–515.

[32] D. Q. Tran, M. L. Dung and V. H. Nguyen, *Extragradient algorithms extended to equilibrium problems*, Optimization **57** (2008), 749–776.

[33] J. A. Van Tiel, *Convex Analysis: An Introductory Text*, Wiley, New York, 1984.

[34] P. T. Vuong, J. J. Strodiot and V. H. Nguyen, *Extragradient methods and linesearch algorithms for solving Ky Fan inequalities and fixed point problems*, J. Optim. Theory Appl. **155** (2012), 605–627.

[35] P. T. Vuong, J. J. Strodiot and V. H. Nguyen, *On extragradient-viscosity methods for solving equilibrium and fixed point problems in a Hilbert space*, Optimization **64** (2013), 429–451.

[36] N. Wairojjana, N. Pholasa, C. Khunpanuk and N. Pakkaranang, *Accelerated strongly convergent extragradient algorithms to solve variational inequalities and fixed point problems in real Hilbert spaces*, Nonlinear Funct. Anal. Appl. **29** (2024), 307–332.

[37] S. Xu and S. Li, *A strongly convergent alternated inertial algorithm for solving equilibrium problems*, J. Optim. Theory Appl. **206** (2025), 520–542.

[38] I. Yamada and N. Ogura, *Hybrid steepest descent method for variational inequality problem over the fixed point set of certain quasi-nonexpansive mappings*, Numer. Funct. Anal. Optim. **25** (2005), 619–655.

[39] Y. Yao, O. S. Iyiola and Y. Shehu, *Subgradient extragradient method with double inertial steps for variational inequalities*, J. Sci. Comput. **90** (2022): Article ID 2.

[40] X. Zhao, M. A. Köbis, Y. Yao and J.-C. Yao, *A projected subgradient method for nondifferentiable quasiconvex multiobjective optimization problems*, J. Optim. Theory Appl. **190** (2021), 82–107.

Manuscript received March 21, 2025
revised October 2, 2025

N. Pholasa
School of Science, University of Phayao, Phayao 56000, Thailand
E-mail address: `nattawut_math@hotmail.com`

N. Pakkaranang
Mathematics and Computing Science Program, Faculty of Science and Technology, Phetchabun Rajabhat University, Phetchabun 67000, Thailand
E-mail address: `nuttapol.pak@pcru.ac.th`

สำเนาถูกต้อง

(นายณัฐพล ปักการะนัง)