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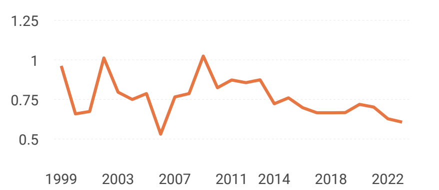

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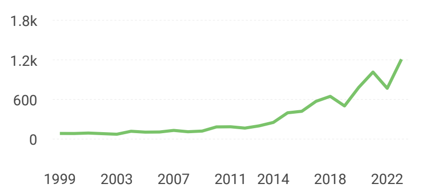

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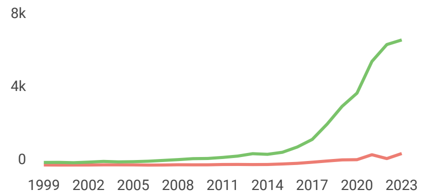

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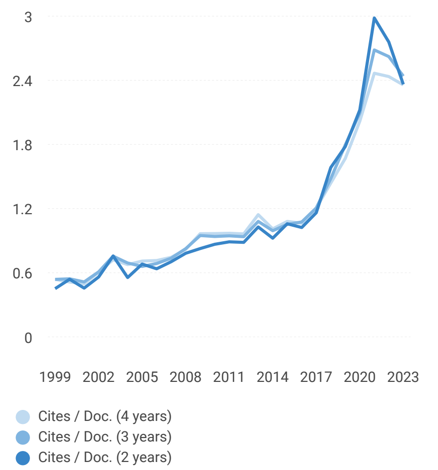

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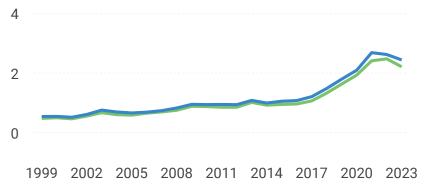

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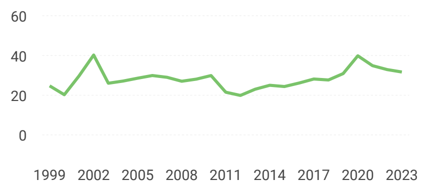

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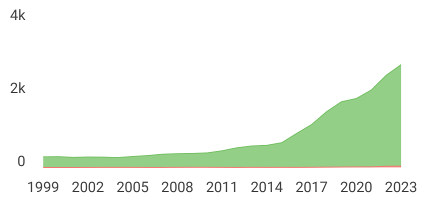

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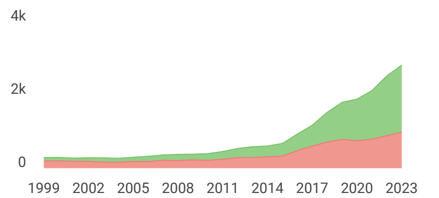

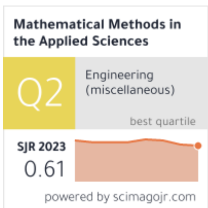

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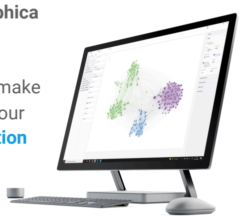

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Double inertial extragradient algorithms for solving variational inequality problems with convergence analysis

Nuttapol Pakkaranang

First published: 30 April 2024 **https://doi.org/10.1002/mma.10147**

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In this paper, we introduce a novel dual inertial Tseng's extragradient method for solving variational inequality problems in real Hilbert spaces, particularly those involving pseudomonotone and Lipschitz continuous operators. Our secondary method incorporates variable step-size, updated at each iteration based on some previous iterates. A notable advantage of these algorithms is their ability to operate without prior knowledge of Lipschitz-type constants and without the need for any line-search procedure. We establish the convergence theorem of the proposed algorithms under mild assumptions. To illustrate the numerical behavior of the algorithms and to make comparisons with other methods, we conduct several numerical experiments. The results of these evaluations are showcased and thoroughly examined to exemplify the practical significance and effectiveness of the proposed methods.

CONFLICTS OF INTEREST STATEMENT

This work does not have any conflicts of interest.

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Mathematical Methods in the Applied Sciences

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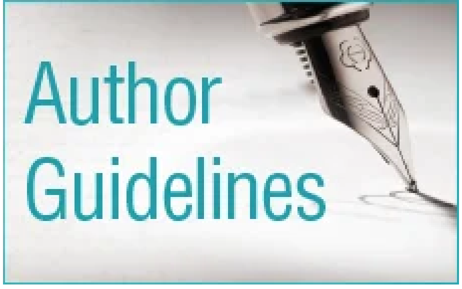

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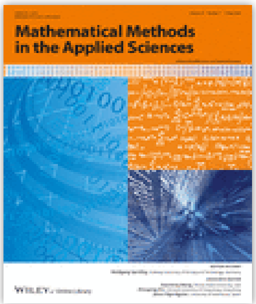

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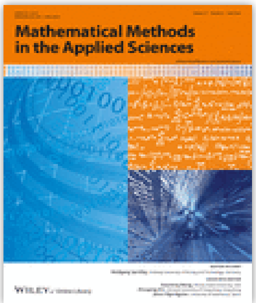

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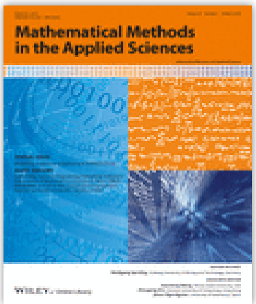

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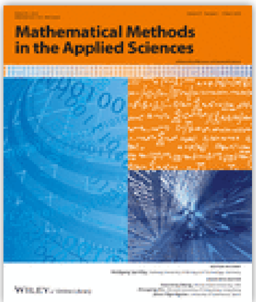

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RESEARCH ARTICLE

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Double inertial extragradient algorithms for solving variational inequality problems with convergence analysis

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Communicated by: J. Vigo-Aguiar

Funding information
Office of the Permanent Secretary, Ministry of Higher Education, Science, Research and Innovation (OPS MHESI), Thailand Science Research and Innovation (TSRI) and Phetchabun Rajabhat University, Grant/Award Number: RGNS65-168

In this paper, we introduce a novel dual inertial Tseng's extragradient method for solving variational inequality problems in real Hilbert spaces, particularly those involving pseudomonotone and Lipschitz continuous operators. Our secondary method incorporates variable step-size, updated at each iteration based on some previous iterates. A notable advantage of these algorithms is their ability to operate without prior knowledge of Lipschitz-type constants and without the need for any line-search procedure. We establish the convergence theorem of the proposed algorithms under mild assumptions. To illustrate the numerical behavior of the algorithms and to make comparisons with other methods, we conduct several numerical experiments. The results of these evaluations are showcased and thoroughly examined to exemplify the practical significance and effectiveness of the proposed methods.

KEYWORDS
Lipschitz continuity, Pseudomonotone mapping, Tseng's extragradient method, Variational inequalities, Weak convergence theorem

MSC CLASSIFICATION
47J25, 47H09, 47H06, 47J05

1 | INTRODUCTION

Consider a real Hilbert space $\mathcal{E}$, equipped with an inner product $\langle \cdot, \cdot \rangle$, and the associated norm $\| \cdot \|$. In this context, $\mathcal{D}$ is defined as a nonempty, closed, and convex subset of $\mathcal{E}$.

This study introduces novel iterative methods designed to address approximate solutions to the variational inequality problem, abbreviated as VIP, within the framework of a real Hilbert space. Let $\mathcal{P} : \mathcal{E} \to \mathcal{E}$ represent an operator, and we define the VIP for $\mathcal{P}$ on the set $\mathcal{D}$ as follows, with reference to earlier studies [1, 2]:

$$\text{Find } u^* \in \mathcal{D} \text{ such that } \langle \mathcal{P}(u^*), v - u^* \rangle \geqslant 0 \text{ for all } v \in \mathcal{D}. \tag{VIP}$$

To initiate the convergence study of the proposed methods, we assume the following conditions:

- **(c1)**: The solution set for a VIP is nonempty, represented as Ω.
- **(c2)**: An operator $\mathcal{P} : \mathcal{E} \to \mathcal{E}$ is pseudomonotone if it satisfies the pseudomonotonicity condition defined as follows:

$$\langle \mathcal{P}(u_1), u_2 - u_1 \rangle \geqslant 0 \Rightarrow \langle \mathcal{P}(u_2), u_1 - u_2 \rangle \leqslant 0 \text{ for all } u_1, u_2 \in \mathcal{D}. \tag{PM}$$

- **(c3)**: Furthermore, an operator $\mathcal{P} : \mathcal{E} \to \mathcal{E}$ is Lipschitz continuous if it has a positive constant $L > 0$ that satisfies the Lipschitz continuity requirement stated as follows:

$$\|\mathcal{P}(u_1) - \mathcal{P}(u_2)\| \leqslant L\|u_1 - u_2\| \text{ for all } u_1, u_2 \in \mathcal{D}. \tag{LC}$$

- **(c4)**: Subsequently, a mapping $\mathcal{P} : \mathcal{E} \to \mathcal{E}$ is considered sequentially weakly continuous if, for any sequence $\{u_k\}$ that weakly converges to u, the sequence $\{\mathcal{P}(u_k)\}$ also weakly converges to $\mathcal{P}(u)$.

The mathematical model of the VIP encompasses a wide range of mathematical problems, such as optimization, optimal control, partial differential equations, mechanics, finance, and mathematical programming. For a comprehensive overview, please refer to previous research [3–6]. The VIP holds a lot of importance in the field of applied sciences. Numerous researchers have dedicated their efforts not only to exploring theoretical aspects related to the existence and stability of solutions but also to devising iterative methods for solving VIPs. Additionally, projection methods have emerged as crucial tools for approximating numerical solutions to these inequalities, with researchers introducing various variations of such approaches to efficiently address these challenging problems (further details are available in earlier studies [7–16]) and others in prior research [17–27].

Almost all solutions methods developed to address the challenge posed by Equation (VIP) revolve around the core concept of projecting onto the feasible set $\mathcal{D}$. Pioneers in this field, such as Korpelevich [9] and Antipin [28], devised the extragradient methods. Let us consider the following mathematical formulation represented as Equation (1.1):

$$\begin{cases} u_0 \in \mathcal{D}, \\ v_k = P_{\mathcal{D}}[u_k - \gamma\mathcal{P}(u_k)], \\ u_{k+1} = P_{\mathcal{D}}[u_k - \gamma\mathcal{P}(v_k)]. \end{cases} \tag{1.1}$$

It is critical to underline that the parameter γ must satisfy the condition $0 < \gamma < \frac{1}{L}$ for this approach to be effective. During each iteration, this method employs a dual projection procedure onto the set $\mathcal{D}$. It is crucial to understand that the computational efficiency of this technique may be impacted, especially when dealing with a complex structure within the feasible set $\mathcal{D}$. In response to this constraint, we provide here multiple developed approaches for solution methods. First, we provide the subgradient extragradient approach as described by Censor et al. [7]. This approach is expressed through the following equations:

$$\begin{cases} u_0 \in \mathcal{D}, \\ v_k = P_{\mathcal{D}}[u_k - \gamma\mathcal{P}(u_k)], \\ u_{k+1} = P_{T_k}[u_k - \gamma\mathcal{P}(v_k)]. \end{cases} \tag{1.2}$$

It is important to note here that $0 < \gamma < \frac{1}{L}$, and the set T_k is defined as follows:

$$T_k = \{z \in \mathcal{E} : \langle u_k - \gamma\mathcal{P}(u_k) - v_k, z - v_k\rangle \leqslant 0\}.$$

The primary objective of this study to provide the improvement of Tseng's extragradient method [11], which employs only one projection in each iteration. This approach is represented by the following formulation:

$$\begin{cases} u_0 \in \mathcal{D}, \\ v_k = P_{\mathcal{D}}[u_k - \gamma\mathcal{P}(u_k)], \\ u_{k+1} = v_k - \gamma[\mathcal{P}(v_k) - \mathcal{P}(u_k)], \end{cases} \tag{1.3}$$

It is crucial to emphasize that $0 < \gamma < \frac{1}{L}$. However, it is important to note that these approaches have some key drawbacks. Firstly, they rely on a fixed constant step-size rule, dependent on the mapping's Lipschitz constant, producing an iteratively weakly convergent sequence. Determining the Lipschitz constant is often challenging or computationally expensive. Secondly, from a computational standpoint, adhering to a set step-size limitation can impact the method's efficiency and convergence rate.

The following natural question has been raised:

"Is it feasible to develop new double inertial Tseng's extragradient-type methods, incorporating a combination of monotonic and non-monotonic variable step-size rules, to effectively address variational inequalities while ensuring weak convergence?"

We provide a positive answer to the prior question: Tseng's extragradient method generates a weakly convergent iterative sequence while a fixed, monotone, and nonmonotone step-size rule is used. The primary objective of this study is to provide an improved version of Tseng's extragradient method, strengthened with double inertial extrapolation steps. The approach we propose has been developed for addressing variational inequalities in Hilbert spaces, particularly when the operator exhibits pseudomonotonic characteristics and conforms to the Lipschitz condition. We provide three distinct variants of Tseng's extragradient-type methods, each featuring corresponding theorems guaranteeing weak convergence. Our approach is inspired by both the projection method [11] and the inertial method [29]. Notably, our method only requires a single projection onto the feasible set $\mathcal{D}$ per iteration. We conduct an extensive investigation to conclusively demonstrate that, with specifically defined control parameters, the iterative sequences generated by our proposed approaches exhibit apparent weak convergence towards a solution. Additionally, we bolster our findings with practical examples that showcase the computational performance of our new methods compared with established ones. In recent years, algorithms incorporating inertial extrapolation steps have garnered attention for their ability to significantly accelerate convergence. When these procedures are integrated into algorithms, they consistently demonstrate their potential to reduce the number of iterations and, consequently, CPU time when solving various problem sets.

In this context, the primary goal is to develop an inertial-type technique based on the framework outlined in Tseng [11]. This method has been carefully designed to improve the convergence rate of iterative sequences. Notably, this method draws inspiration from second-order dynamical systems known as "heavy friction balls", as first presented by Polyak in [29]. The utilization of knowledge gathered from two previous iterations to inform the construction of the current one characterizes this method. Additionally, our proposed method features a notable aspect: the ability to incorporate an inertial factor in the range $[0, 1]$. This distinctive characteristic sets our research apart from previous studies that did not delve into such details. In short, our research advances our understanding of inertial approaches to solving VIPs.

The overall format of the paper is as follows: In Section 2, we delve into fundamental concepts and lay the groundwork for our subsequent main results. Section 3 introduces our proposed methods, supported by well-established convergence theorems that validate their effectiveness. Finally, Section 4 provides numerical data demonstrating the convergence and overall efficacy of the proposed methods.

2 | PRELIMINARIES

This section will explore fundamental concepts, important lemmas, and key definitions. To begin, we define the *metric projection* $P_{\mathcal{D}}(v_1)$ for $v_1 \in \mathcal{E}$ as follows:

$$P_{\mathcal{D}}(v_1) = \arg\min_{v_2 \in \mathcal{D}} \|v_1 - v_2\|.$$

Lemma 2.1 ([30]). *Let $P_{\mathcal{D}} : \mathcal{E} \to \mathcal{D}$ be a metric projection. Then, the following conditions are satisfied.*

(i) $v_3 = P_{\mathcal{D}}(v_1)$ *if and only if*

$$\langle v_1 - v_3, v_2 - v_3 \rangle \leqslant 0, \ \forall v_2 \in \mathcal{D}.$$

(ii)

$$\|v_1 - P_{\mathcal{D}}(v_2)\|^2 + \|P_{\mathcal{D}}(v_2) - v_2\|^2 \leqslant \|v_1 - v_2\|^2, \ \forall v_1 \in \mathcal{D}, v_2 \in \mathcal{E}.$$

(iii)

$$\|v_1 - P_{\mathcal{D}}(v_1)\| \leqslant \|v_1 - v_2\| \ \forall v_2 \in \mathcal{D}, v_1 \in \mathcal{E}.$$

Lemma 2.2 ([30]). *Consider any v_1 and v_2 in $\mathcal{E}$ and $\ell \in \mathbb{R}$. The following conditions are stated:*

1. *The square of the norm of a linear combination is expressed as follows:*

$$\|\ell v_1 + (1 - \ell)v_2\|^2 = \ell\|v_1\|^2 + (1 - \ell)\|v_2\|^2 - \ell(1 - \ell)\|v_1 - v_2\|^2.$$

2. *For the sum of two vectors, the following inequality holds:*

$$\|v_1 + v_2\|^2 \leqslant \|v_1\|^2 + 2\langle v_2, v_1 + v_2 \rangle.$$

Lemma 2.3 ([31]). *Assume three sequences denoted as b_k, c_k, and d_k, all of which correspond to the interval $[0, +\infty)$. For any $k \geqslant 1$, these sequences satisfy the following inequality:*

$$b_{k+1} \leqslant b_k + d_k(b_k - b_{k-1}) + c_k.$$

Additionally, we are given that the series $\sum_{k=1}^{+\infty} c_k$ converges, denoted as $\sum_{k=1}^{+\infty} c_k < +\infty$. Under these conditions, there exists a real number d such that $0 \leqslant d_k \leqslant d < 1$ for every k in the set of natural numbers. Consequently,

(i) *The sum of all positive parts of the differences between successive terms in the sequence b_k converges, denoted as $\sum_{k=1}^{+\infty} [b_k - b_{k-1}]_+ < +\infty$. Here, $[t]_+ := \max\{t, 0\}$.*
(ii) *Additionally, there exists a real number b^* within the range $[0, +\infty)$ such that the limit of the sequence $\{b_k\}$ as k approaches infinity is equal to b^*.*

Lemma 2.4 ([32]). *Take a nonempty subset $\mathcal{D}$ of a Hilbert space $\mathcal{E}$ and a sequence denoted by $\{u_k\}$ in $\mathcal{E}$. This sequence must satisfy the following two requirements:*

(i) *For every $u \in \mathcal{D}$, $\lim_{k\to+\infty} \|u_k - u\|$ exists.*
(ii) *Every sequentially weak cluster point of $\{u_k\}$ is in $\mathcal{D}$.*

Then, $\{u_k\}$ converges weakly to a point in $\mathcal{D}$.

3 MAIN RESULTS

In this section, introducing improved versions of Tseng's extragradient method for addressing VIPs, we employ a numerical iterative method aimed at enhancing the convergence rate of the iterative sequence. This involves incorporating two convex minimization problems with a two-step inertial scheme. The process commences from two given starting points and systematically updates the outcome at each step using predefined formulas. Termination occurs when a specified condition is met, specifically when $v_k = w_k$. Subsequent sections will delve into a comprehensive discussion of the algorithmic procedures and conduct a thorough examination of its convergence characteristics.

Algorithm 1 Double Inertial Tseng's Extragradient Method With Fixed Step-size Rule

Input: Starting iterates $u_{-1}, u_0 \in \mathcal{E}$, $0 < \gamma_k < \frac{1}{L}$, $\alpha \in [0, 1)$, $\beta \in [0, 1]$, $\zeta \in (0, 1)$, and convex set $\mathcal{D}$.
Output: Solution u_{k+1}
for $k = 1, 2, \ldots$ **do**
  Calculate inertial iterations:
    $s_k = u_k + \alpha(u_k - u_{k-1})$,
    $w_k = u_k + \beta(u_k - u_{k-1})$.
  Solve the subproblem for v_k as follows:
    $v_k = P_{\mathcal{D}}(w_k - \gamma_k \mathcal{P}(w_k))$.
  Determine the value of q_k as follows:
    $q_k = v_k + \gamma_k [\mathcal{P}(w_k) - \mathcal{P}(v_k)]$.
  Update the iteration u_{k+1} as follows:
    $u_{k+1} = (1 - \zeta)s_k + \zeta q_k$.
  if $v_k = w_k$ **then**
    break ▷ Convergence has been attained.

Algorithm 2 Double Inertial Tseng's Extragradient Method With Monotone Step-size Rule

Input: Initial iterations $u_{-1}, u_0 \in \mathcal{E},\ \gamma_0 > 0,\ \alpha \in [0,1),\ \beta \in [0,1],\ \mu \in (0,1),\ \zeta \in (0,1)$, and convex set $\mathcal{D}$.
Output: Solution u_{k+1}
for $k = 1, 2, \dots$ **do**
  Iteratively compute inertial iterations:
    $s_k = u_k + \alpha(u_k - u_{k-1})$,
    $w_k = u_k + \beta(u_k - u_{k-1})$.
  Solve the subproblem for v_k as follows:
    $v_k = P_{\mathcal{D}}(w_k - \gamma_k \mathcal{P}(w_k))$.
  Determine the value of q_k:
    $q_k = v_k + \gamma_k [\mathcal{P}(w_k) - \mathcal{P}(v_k)]$
  Update the iteration u_{k+1}:
    $u_{k+1} = (1-\zeta)s_k + \zeta q_k$.
  The formula for γ_{k+1} is as follows:

$$\gamma_{k+1} = \begin{cases} \min\left\{\gamma_k, \frac{\mu\|w_k - v_k\|}{\|\mathcal{P}(w_k) - \mathcal{P}(v_k)\|}\right\} & \text{if} \quad \mathcal{P}(w_k) - \mathcal{P}(v_k) \neq 0, \\ \gamma_k \text{otherwise.} \end{cases} \tag{3.1}$$

  if $v_k = w_k$ **then**
    break ▷ Convergence has been attained.

Lemma 3.1. *A sequence of step-sizes $\{\gamma_k\}$ generated by (3.1) converges to a fixed positive value $\gamma > 0$.*

Proof. The sequence $\{\gamma_k\}$ is clearly monotone and non-increasing. Considering the Lipschitz continuity of the mapping $\mathcal{P}$, we can demonstrate the existence of a positive constant $L > 0$ such that:

$$\begin{aligned} \frac{\mu\|w_k - v_k\|}{\|\mathcal{P}(w_k) - \mathcal{P}(v_k)\|} &\geqslant \frac{\mu\|w_k - v_k\|}{L\|w_k - v_k\|} \\ &\geqslant \frac{\mu}{L}. \end{aligned} \tag{3.2}$$

This inequality implies that the sequence $\{\gamma_k\}$ has a lower bound, namely, $\min\left\{\frac{\mu}{L}, \gamma_0\right\}$. Additionally, there exists a positive number $\gamma > 0$ such that $\lim_{k\to\infty}\gamma_k = \gamma$. □

It is crucial to emphasize that the chosen step-size rule should align with the nature of the problem and the algorithm's parameters. This consideration is especially significant as non-monotone rules may introduce additional computational challenges. It is noteworthy that the selection of a step-size rule relies on the features of the VIP and the characteristics of the optimization algorithm. Nonmonotone step-size rules may entail substantial trade-offs, such as increased memory requirements and computational complexities, necessitating careful consideration. Consequently, determining the appropriate step-size rule should be based on a comprehensive evaluation of both the inherent characteristics of the problem and the specific requirements of the algorithm.

Note 1: Algorithm 1 employs a monotone step-size rule, but it can be adapted to use a non-monotone step-size rule. Nonmonotone step-size rules adjust the step-size based on prior iteration information, providing more adaptability and exploration during optimization, to incorporate a nonmonotone step-size rule into the algorithm.

Choose a nonnegative real sequence denoted as $\{p_k\}$ with the requirement that $\sum_{k=1}^{+\infty} p_k < +\infty$. The formula for γ_{k+1} is as follows:

$$\gamma_{k+1} = \begin{cases} \min\left\{\gamma_k + p_k, \frac{\mu\|w_k - v_k\|}{\|\mathcal{P}(w_k) - \mathcal{P}(v_k)\|}\right\} & \text{if } \mathcal{P}(w_k) - \mathcal{P}(v_k) \neq 0, \\ \gamma_k + p_k & \text{otherwise.} \end{cases} \tag{3.3}$$

Also, choose another nonnegative real sequence $\{p_k\}$ defined as $p_k = 1 + d_k$, where $\sum_{k=1}^{+\infty} d_k < +\infty$. The formula for γ_{k+1} is as follows:

$$\gamma_{k+1} = \begin{cases} \min\left\{\gamma_k \cdot p_k, \frac{\mu\|w_k - v_k\|}{\|\mathcal{P}(w_k) - \mathcal{P}(v_k)\|}\right\} & \text{if } \mathcal{P}(w_k) - \mathcal{P}(v_k) \neq 0, \\ \gamma_k \cdot p_k & \text{otherwise.} \end{cases} \tag{3.4}$$

Lemma 3.2. *The sequence $\{\gamma_k\}$ resulting from (3.3) converges to a fixed value $\gamma > 0$ and satisfies the following inequality:*

$$\min\left\{\frac{\mu}{L}, \gamma_0\right\} \leqslant \gamma_k \leqslant \gamma_0 + P, \ \textit{where}\ P = \sum_{k=1}^{+\infty} p_k.$$

Proof. The Lipschitz continuity of the mapping $\mathcal{P}$ yields a fixed constant $L > 0$. Consider the formula $\mathcal{P}(w_k) - \mathcal{P}(v_k) \neq 0$, which allows us to write the following:

$$\begin{aligned} \frac{\mu\|w_k - v_k\|}{\|\mathcal{P}(w_k) - \mathcal{P}(v_k)\|} &\geqslant \frac{\mu\|w_k - v_k\|}{L\|w_k - v_k\|} \\ &\geqslant \frac{\mu}{L}. \end{aligned} \tag{3.5}$$

Applying an inductive method based on γ_{k+1}, we prove the following:

$$\min\left\{\frac{\mu}{L}, \gamma_0\right\} \leqslant \gamma_k \leqslant \gamma_0 + P.$$

Now, set up the notations $[\gamma_{k+1} - \gamma_k]^+ = \max\{0, \gamma_{k+1} - \gamma_k\}$ and $[\gamma_{k+1} - \gamma_k]^- = \max\{0, -(\gamma_{k+1} - \gamma_k)\}$. Using $\{\gamma_k\}$, we can write the following:

$$\sum_{k=1}^{+\infty} (\gamma_{k+1} - \gamma_k)^+ = \sum_{k=1}^{+\infty} \max\{0, \gamma_{k+1} - \gamma_k\} \leqslant P < +\infty. \tag{3.6}$$

This indicates that the series $\sum_{k=1}^{+\infty} (\gamma_{k+1} - \gamma_k)^+$ is convergent. Now, examine the convergence of $\sum_{k=1}^{+\infty} (\gamma_{k+1} - \gamma_k)^-$. Let

$$\sum_{k=1}^{+\infty} (\gamma_{k+1} - \gamma_k)^- = +\infty.$$

Take advantage of the fact that:

$$\gamma_{k+1} - \gamma_k = (\gamma_{k+1} - \gamma_k)^+ - (\gamma_{k+1} - \gamma_k)^-.$$

This enables us to write the following:

$$\gamma_{k+1} - \gamma_0 = \sum_{k=0}^{k} (\gamma_{k+1} - \gamma_k) = \sum_{k=0}^{k} (\gamma_{k+1} - \gamma_k)^+ - \sum_{k=0}^{k} (\gamma_{k+1} - \gamma_k)^-. \tag{3.7}$$

Considering the limit as $k \to +\infty$ in (3.7), we obtain $\gamma_k \to -\infty$ as $k \to \infty$. This contradicts our previous findings. As a result of the convergence of the series $\sum_{k=0}^{k} (\gamma_{k+1} - \gamma_k)^+$ and $\sum_{k=0}^{k} (\gamma_{k+1} - \gamma_k)^-$, we may deduce that $\lim_{k\to\infty}\gamma_k = \gamma$. This completes the proof. □

Lemma 3.3. *Consider the mapping $\mathcal{P} : \mathcal{E} \to \mathcal{E}$, which satisfies criteria* ***(c1)–(c4)*** *and a sequence $\{u_k\}$ generated by Algorithm 1. Then, for any $u^* \in \Omega$, the inequality holds:*

$$\|q_k - u^*\|^2 \leqslant \|w_k - u^*\|^2 - \left(1 - \gamma^2 L^2\right) \|w_k - v_k\|^2.$$

Proof. Taking $u^* \in \Omega$ and the definition of q_k, we have the following:

$$\begin{aligned}
&\|q_k - u^*\|^2 \\
&= \|v_k + \gamma[\mathcal{P}(w_k) - \mathcal{P}(v_k)] - u^*\|^2 \\
&= \|v_k - u^*|^2 + \gamma^2\|\mathcal{P}(w_k) - \mathcal{P}(v_k)\|^2 + 2\gamma\langle v_k - u^*, \mathcal{P}(w_k) - \mathcal{P}(v_k)\rangle \\
&= \|v_k + w_k - w_k - u^*\|^2 + \gamma^2\|\mathcal{P}(w_k) - \mathcal{P}(v_k)\|^2 \\
&\quad + 2\gamma\langle v_k - u^*, \mathcal{P}(w_k) - \mathcal{P}(v_k)\rangle \\
&= \|v_k - w_k\|^2 + \|w_k - u^*\|^2 + 2\langle v_k - w_k, w_k - u^*\rangle \\
&\quad + \gamma^2\|\mathcal{P}(w_k) - \mathcal{P}(v_k)\|^2 + 2\gamma\langle v_k - u^*, \mathcal{P}(w_k) - \mathcal{P}(v_k)\rangle \\
&= \|w_k - u^*\|^2 + \|v_k - w_k\|^2 + 2\gamma\langle \mathcal{P}(w_k), u^* - v_k\rangle - 2\langle w_k - v_k, w_k - v_k\rangle \\
&\quad + \gamma^2\|\mathcal{P}(w_k) - \mathcal{P}(v_k)\|^2 - 2\gamma\langle \mathcal{P}(w_k) - \mathcal{P}(v_k), u^* - v_k\rangle.
\end{aligned} \tag{3.8}$$

Given that

$$v_k = P_{\mathcal{D}}[w_k - \gamma\mathcal{P}(w_k)],$$

and it further implies that

$$\langle w_k - \gamma\mathcal{P}(w_k) - v_k, v - v_k\rangle \leqslant 0, \ \forall v \in \mathcal{D}. \tag{3.9}$$

As a result, we can conclude the following:

$$\langle w_k - v_k, u^* - v_k\rangle \leqslant \gamma\langle \mathcal{P}(w_k), u^* - v_k\rangle. \tag{3.10}$$

Combining (3.8) with (3.10) gives the following result:

$$\begin{aligned}
\|q_k - u^*\|^2 &\leqslant \|w_k - u^*\|^2 + \|v_k - w_k\|^2 + 2\gamma\langle \mathcal{P}(w_k), u^* - v_k\rangle - 2\langle w_k - v_k, w_k - v_k\rangle \\
&\quad + \gamma^2\|\mathcal{P}(w_k) - \mathcal{P}(v_k)\|^2 - 2\gamma\langle \mathcal{P}(w_k) - \mathcal{P}(v_k), u^* - v_k\rangle \\
&= \|w_k - u^*\|^2 - \|w_k - v_k\|^2 + \gamma^2\|\mathcal{P}(w_k) - \mathcal{P}(v_k)\|^2 - 2\gamma\langle \mathcal{P}(v_k), v_k - u^*\rangle.
\end{aligned} \tag{3.11}$$

As u^* is the solution to the problem (VIP), it follows that

$$\langle \mathcal{P}(u^*), v - u^*\rangle \geqslant 0, \ \forall v \in \mathcal{D}.$$

In addition, due to the pseudomonotone property of the mapping $\mathcal{P}$ on $\mathcal{D}$, we can conclude the following:

$$\langle \mathcal{P}(v), v - u^*\rangle \geqslant 0, \ \forall v \in \mathcal{D}.$$

By inserting $v = v_k \in \mathcal{D}$, we obtain the following:

$$\langle \mathcal{P}(v_k), v_k - u^*\rangle \geqslant 0. \tag{3.12}$$

Combining expressions (3.11) and (3.12), we conclude the following:

$$\begin{aligned}
\|q_k - u^*\|^2 &\leqslant \|w_k - u^*\|^2 - \|w_k - v_k\|^2 + \gamma^2 L^2\|w_k - v_k\|^2 \\
&= \|w_k - u^*\|^2 - \left(1 - \gamma^2 L^2\right)\|w_k - v_k\|^2.
\end{aligned} \tag{3.13}$$

This completes the proof of lemma. □

Lemma 3.4. *Let $\mathcal{P} : \mathcal{E} \to \mathcal{E}$ be a mapping that satisfies the conditions* **(c1)–(c4)***. Moreover, let $\{u_k\}$ be a sequence generated by Algorithm 2. For any $u^* \in \Omega$, the following inequality holds:*

$$\|q_k - u^*\|^2 \leqslant \|w_k - u^*\|^2 - \left(1 - \mu^2 \frac{\gamma_k^2}{\gamma_{k+1}^2}\right) \|w_k - v_k\|^2.$$

Proof. Since $u^* \in \Omega$ and using the definition of q_k, we obtain

$$\begin{aligned}
\|q_k - u^*|^2 &= \|v_k + \gamma_k[\mathcal{P}(w_k) - \mathcal{P}(v_k)] - u^*\|^2 \\
&= \|v_k - u^*\|^2 + \gamma_k^2\|\mathcal{P}(w_k) - \mathcal{P}(v_k)\|^2 + 2\gamma_k\langle v_k - u^*, \mathcal{P}(w_k) - \mathcal{P}(v_k)\rangle \\
&= \|v_k + w_k - w_k - u^*\|^2 + \gamma_k^2\|\mathcal{P}(w_k) - \mathcal{P}(v_k)\|^2 \\
&\quad + 2\gamma_k\langle v_k - u^*, \mathcal{P}(w_k) - \mathcal{P}(v_k)\rangle \\
&= \|v_k - w_k\|^2 + \|w_k - u^*\|^2 + 2\langle v_k - w_k, w_k - u^*\rangle \\
&\quad + \gamma_k^2\|\mathcal{P}(w_k) - \mathcal{P}(v_k)\|^2 \\
&\quad + 2\gamma_k\langle v_k - u^*, \mathcal{P}(w_k) - \mathcal{P}(v_k)\rangle \\
&= \|w_k - u^*\|^2 + \|v_k - w_k\|^2 \\
&\quad + 2\langle v_k - w_k, v_k - u^*\rangle + 2\langle v_k - w_k, w_k - v_k\rangle \\
&\quad + \gamma_k^2\|\mathcal{P}(w_k) - \mathcal{P}(v_k)\|^2 \\
&\quad + 2\gamma_k\langle v_k - u^*, \mathcal{P}(w_k) - \mathcal{P}(v_k)\rangle.
\end{aligned} \tag{3.14}$$

By using $v_k = P_{\mathcal{D}}[w_k - \gamma_k\mathcal{P}(w_k)]$, we have the following:

$$\langle w_k - \gamma_k\mathcal{P}(w_k) - v_k, v - v_k\rangle \leqslant 0, \ \forall v \in \mathcal{D}. \tag{3.15}$$

This implies that

$$\langle w_k - v_k, u^* - v_k\rangle \leqslant \gamma_k\langle \mathcal{P}(w_k), u^* - v_k\rangle. \tag{3.16}$$

Combining Equations (3.14) and (3.16), we obtain the following:

$$\begin{aligned}
\|q_k - u^*\|^2 \leqslant\ & \|w_k - u^*\|^2 + \|v_k - w_k\|^2 \\
& + 2\gamma_k\langle \mathcal{P}(w_k), u^* - v_k\rangle - 2\langle w_k - v_k, w_k - v_k\rangle \\
& + \gamma_k^2\|\mathcal{P}(w_k) - \mathcal{P}(v_k)\|^2 \\
& - 2\gamma_k\langle \mathcal{P}(w_k) - \mathcal{P}(v_k), u^* - v_k\rangle \\
=\ & \|w_k - u^*\|^2 - \|w_k - v_k\|^2 \\
& + \gamma_k^2\|\mathcal{P}(w_k) - \mathcal{P}(v_k)\|^2 \\
& - 2\gamma_k\langle \mathcal{P}(v_k), v_k - u^*\rangle.
\end{aligned} \tag{3.17}$$

It is given that u^* is the solution to the problem (VIP), we have the following:

$$\langle \mathcal{P}(u^*), v - u^*\rangle \geqslant 0, \ \forall v \in \mathcal{D}.$$

Using the pseudomonotonicity feature of the mapping $\mathcal{P}$ on $\mathcal{D}$, we may deduce the following:

$$\langle \mathcal{P}(v), v - u^*\rangle \geqslant 0, \ \forall v \in \mathcal{D}.$$

By inserting $v = v_k \in \mathcal{D}$, we obtain the following:

$$\langle \mathcal{P}(v_k), v_k - u^*\rangle \geqslant 0. \tag{3.18}$$

Considering Equations (3.17) and (3.18), we derive the following:

$$\begin{aligned}\|q_k - u^*\|^2 &\leqslant \|w_k - u^*\|^2 - \|w_k - v_k\|^2 + \mu^2 \frac{\gamma_k^2}{\gamma_{k+1}^2}\|w_k - v_k\|^2 \\ &= \|w_k - u^*\|^2 - \left(1 - \mu^2 \frac{\gamma_k^2}{\gamma_{k+1}^2}\right)\|w_k - v_k\|^2.\end{aligned} \tag{3.19}$$

□

Theorem 3.5. *Assume $\{u_k\}$ represents a sequence created by Algorithm 1 under conditions* ***(c1)–(c4)*** *and with*

$$\zeta \leqslant \frac{1}{2} \text{ and } \zeta \leqslant \frac{1+\alpha^2-2\alpha}{(1+2\alpha^2-\alpha)+\beta(1+\beta)}.$$

Consequently, it can be inferred that the sequence $\{u_k\}$ weakly converges to the limit $u^* \in \Omega$.

Proof. By the use of Lemma 3.4, we have

$$\|q_k - u^*\|^2 \leqslant \|w_k - u^*\|^2 - \left(1 - \mu^2 \frac{\gamma_k^2}{\gamma_{k+1}^2}\right)\|w_k - v_k\|^2. \tag{3.20}$$

Since the step-size sequence $\gamma_k \to \gamma$ and there exists a constant number $\epsilon \in (0, 1-\mu^2)$ such that

$$\lim_{k\to\infty}\left(1 - \mu^2 \frac{\gamma_k^2}{\gamma_{k+1}^2}\right) = 1 - \mu^2 > \epsilon > 0.$$

From the above explanation, there exists a fixed number $K_1^* \in \mathbb{N}$ such that

$$\left(1 - \mu^2 \frac{\gamma_k^2}{\gamma_{k+1}^2}\right) > \epsilon > 0, \ \forall k \geqslant K_1^*. \tag{3.21}$$

Furthermore, we have

$$\|q_k - u^*\|^2 \leqslant \|w_k - u^*\|^2, \ \forall k \geqslant K_1^*. \tag{3.22}$$

On the other hand, we have

$$\begin{aligned}\|u_{k+1} - u^*\| &= \|(1-\zeta)s_k + \zeta q_k - u^*\| \\ &= \|(1-\zeta)(s_k - u^*) + \zeta(q_k - u^*)\| \\ &= (1-\zeta)\|s_k - u^*\|^2 + \zeta\|q_k - u^*\|^2 - \zeta(1-\zeta)\|q_k - s_k\|^2.\end{aligned} \tag{3.23}$$

Substituting (3.22) into (3.23), we obtain

$$\|u_{k+1} - u^*\| \leqslant (1-\zeta)\|s_k - u^*\|^2 + \zeta\|w_k - u^*\|^2 - \zeta(1-\zeta)\|q_k - s_k\|^2, \ \forall k \geqslant K_1^*. \tag{3.24}$$

We observe that the update rule for the sequence is given by the following:

$$u_{k+1} = (1-\zeta)s_k + \zeta q_k. \tag{3.25}$$

Moreover, we can deduce from (3.25) that the difference between consecutive iterates q_k and s_k can be expressed as follows:

$$q_k - s_k = \frac{1}{\zeta}(u_{k+1} - s_k). \tag{3.26}$$

Substituting (3.26) into (3.24), we obtain

$$\|u_{k+1} - u^*\| \leqslant (1-\zeta)\|s_k - u^*\|^2 + \zeta\|w_k - u^*\|^2 - \frac{(1-\zeta)}{\zeta}\|u_{k+1} - s_k\|^2. \tag{3.27}$$

By substituting the value of w_k from Algorithm 1 and utilizing Lemma 2.2 (i), we can derive the following expressions. First, we find the value of $\|w_k - u^*\|^2$ as follows:

$$\begin{aligned} \|w_k - u^*\|^2 &= \|u_k + \beta(u_k - u_{k-1}) - u^*\|^2 \\ &= \|(1+\beta)(u_k - u^*) - \beta(u_{k-1} - u^*)\|^2 \\ &= (1+\beta)\|u_k - u^*\|^2 - \beta\|u_{k-1} - u^*\|^2 + \beta(1+\beta)\|u_k - u_{k-1}\|^2. \end{aligned} \tag{3.28}$$

Next, we determine the value of $\|s_k - u^*\|^2$ as follows:

$$\begin{aligned} \|s_k - u^*\|^2 &= \|u_k + \alpha(u_k - u_{k-1}) - u^*\|^2 \\ &= \|(1+\alpha)(u_k - u^*) - \alpha(u_{k-1} - u^*)\|^2 \\ &= (1+\alpha)\|u_k - u^*\|^2 - \alpha\|u_{k-1} - u^*\|^2 + \alpha(1+\alpha)\|u_k - u_{k-1}\|^2. \end{aligned} \tag{3.29}$$

These formulations offer valuable insights into the connection between $\|w_k - u^*\|$ and $\|s_k - u^*\|$ and how they are influenced by the parameters α and β, along with the variances between consecutive iterates u_k and u_{k-1}. These insights will play a crucial role in our subsequent analysis. By substituting (3.28) and (3.29) into (3.27), we obtain the following inequality:

$$\begin{aligned} \|u_{k+1} - u^*\| \leqslant{}& (1-\zeta)\left[(1+\alpha)\|u_k - u^*\|^2 - \alpha\|u_{k-1} - u^*\|^2 + \alpha(1+\alpha)\|u_k - u_{k-1}\|^2\right] \\ &+ \zeta\left[(1+\beta)\|u_k - u^*\|^2 - \beta\|u_{k-1} - u^*\|^2 + \beta(1+\beta)\|u_k - u_{k-1}\|^2\right] - \frac{(1-\zeta)}{\zeta}\|u_{k+1} - s_k\|^2 \\ \leqslant{}& [(1-\zeta)(1+\alpha) + \zeta(1+\beta)]\,\|u_k - u^*\|^2 - [(1-\zeta)\alpha + \zeta\beta]\,\|u_{k-1} - u^*\|^2 \\ &+ [(1-\zeta)\alpha(1+\alpha) + \zeta\beta(1+\beta)]\,\|u_k - u_{k-1}\|^2 - \frac{(1-\zeta)}{\zeta}\|u_{k+1} - s_k\|^2. \end{aligned} \tag{3.30}$$

By replacing the value of s_k and employing the Cauchy inequality, we obtain the following:

$$\begin{aligned} \|u_{k+1} - s_k\|^2 &= \|u_{k+1} - u_k - \alpha(u_k - u_{k-1})\|^2 \\ &= \|u_{k+1} - u_k\|^2 + \alpha^2\|u_k - u_{k-1}\|^2 - 2\alpha\langle u_{k+1} - u_k, u_k - u_{k-1}\rangle \end{aligned} \tag{3.31}$$

$$\begin{aligned} &\geqslant \|u_{k+1} - u_k\|^2 + \alpha^2\|u_k - u_{k-1}\|^2 - 2\alpha\|u_{k+1} - u_k\|\|u_k - u_{k-1}\| \\ &\geqslant \|u_{k+1} - u_k\|^2 + \alpha^2\|u_k - u_{k-1}\|^2 - \alpha\|u_{k+1} - u_k\|^2 - \alpha\|u_k - u_{k-1}\|^2 \\ &= (1-\alpha)\|u_{k+1} - u_k\|^2 + (\alpha^2 - \alpha)\|u_k - u_{k-1}\|^2. \end{aligned} \tag{3.32}$$

These findings offer a detailed comprehension of the iterative process and the connections among the iterates u_k, u_{k+1}, and the desired solution u^*. These inequalities will play a pivotal role in our subsequent analysis and proofs of convergence. By incorporating (3.32) into (3.30), we infer

$$
\begin{aligned}
\|u_{k+1}-u^*\| \leqslant\ & [(1-\zeta)(1+\alpha)+\zeta(1+\beta)]\,\|u_k-u^*\|^2-[(1-\zeta)\alpha+\zeta\beta]\,\|u_{k-1}-u^*\|^2\\
&+[(1-\zeta)\alpha(1+\alpha)+\zeta\beta(1+\beta)]\,\|u_k-u_{k-1}\|^2\\
&-\frac{(1-\zeta)}{\zeta}\left[(1-\alpha)\|u_{k+1}-u_k\|^2+(\alpha^2-\alpha)\|u_k-u_{k-1}\|^2\right]\\
=\ & [(1-\zeta)(1+\alpha)+\zeta(1+\beta)]\,\|u_k-u^*\|^2-[(1-\zeta)\alpha+\zeta\beta]\,\|u_{k-1}-u^*\|^2\\
&+\left[(1-\zeta)\alpha(1+\alpha)+\zeta\beta(1+\beta)-(\alpha^2-\alpha)\frac{(1-\zeta)}{\zeta}\right]\|u_k-u_{k-1}\|^2\\
&-\frac{(1-\zeta)}{\zeta}(1-\alpha)\|u_{k+1}-u_k\|^2\\
=\ & \|u_k-u^*\|^2+[(1-\zeta)\alpha+\zeta\beta]\,\|u_k-u^*\|^2-[(1-\zeta)\alpha+\zeta\beta]\,\|u_{k-1}-u^*\|^2\\
&+\left[(1-\zeta)\alpha(1+\alpha)+\zeta\beta(1+\beta)-(\alpha^2-\alpha)\frac{(1-\zeta)}{\zeta}\right]\|u_k-u_{k-1}\|^2\\
&-\frac{(1-\zeta)}{\zeta}(1-\alpha)\|u_{k+1}-u_k\|^2\\
=\ & \|u_k-u^*\|^2+[(1-\zeta)\alpha+\zeta\beta]\,\|u_k-u^*\|^2-[(1-\zeta)\alpha+\zeta\beta]\,\|u_{k-1}-u^*\|^2\\
&+\mu\|u_k-u_{k-1}\|^2-\rho\|u_{k+1}-u_k\|^2
\end{aligned}
\tag{3.33}
$$

where

$$\rho=\frac{(1-\zeta)}{\zeta}(1-\alpha)$$

and

$$\mu=\left[(1-\zeta)\alpha(1+\alpha)+\zeta\beta(1+\beta)-(\alpha^2-\alpha)\frac{(1-\zeta)}{\zeta}\right].$$

Subsequently, we substitute

$$\Psi_k=\|u_k-u^*\|^2-[(1-\zeta)\alpha+\zeta\beta]\,\|u_{k-1}-u^*\|^2+\mu\|u_k-u_{k-1}\|^2.$$

Following that, it is necessary to calculate

$$
\begin{aligned}
\Psi_{k+1}-\Psi_k=\ & \|u_{k+1}-u^*\|^2-[(1-\zeta)\alpha+\zeta\beta]\,\|u_k-u^*\|^2+\mu\|u_{k+1}-u_k\|^2\\
&-\|u_k-u^*\|^2+[(1-\zeta)\alpha+\zeta\beta]\,\|u_{k-1}-u^*\|^2-\mu\|u_k-u_{k-1}\|^2\\
\leqslant\ & -(\rho-\mu)\|u_{k+1}-u_k\|^2.
\end{aligned}
\tag{3.34}
$$

Let us calculate

$$
\begin{aligned}
\rho-\mu &= \left[\frac{(1-\zeta)}{\zeta}(1-\alpha)\right]-\left[(1-\zeta)\alpha(1+\alpha)+\zeta\beta(1+\beta)-(\alpha^2-\alpha)\frac{(1-\zeta)}{\zeta}\right]\\
&=\left[\frac{(1-\zeta)}{\zeta}(1-\alpha)\right]-[(1-\zeta)\alpha(1+\alpha)+\zeta\beta(1+\beta)]+\left[(\alpha^2-\alpha)\frac{(1-\zeta)}{\zeta}\right]\\
&=\left[\frac{(1-\zeta)}{\zeta}(1-\alpha)^2-(1-\zeta)\alpha(1+\alpha)-\zeta\beta(1+\beta)\right].
\end{aligned}
\tag{3.35}
$$

Starting with the constraints $\zeta \leqslant \frac{1}{2}$ and $\zeta \leqslant \frac{1+\alpha^2-2\alpha}{(1+2\alpha^2-\alpha)+\beta(1+\beta)}$, let us consider the equation:

$$\zeta \leqslant \frac{1+\alpha^2-2\alpha}{(1+2\alpha^2-\alpha)+\beta(1+\beta)}. \tag{3.36}$$

Considering that $\alpha \in [0, 1)$ and $\beta \in [0, 1]$, we can assert that $(1+2\alpha^2-\alpha)+\beta(1+\beta) > 0$. Simplifying the right-hand side, we derive the following:

$$\zeta\left[(1+2\alpha^2-\alpha)+\beta(1+\beta)\right] \leqslant (1-\alpha)^2. \tag{3.37}$$

Moreover, we have the following:

$$\left[(1+2\alpha^2-\alpha)+\beta(1+\beta)\right] \leqslant \frac{1}{\zeta}(1-\alpha)^2. \tag{3.38}$$

Moreover, we can infer the following:

$$\frac{1}{\zeta}(1-\alpha)^2-(1+2\alpha^2-\alpha)-\beta(1+\beta) \geqslant 0. \tag{3.39}$$

It is evident that $(1+2\alpha^2-\alpha) \geqslant \alpha(1+\alpha)$ for $\alpha \in [0, 1)$, leading to the following:

$$\frac{1-\zeta}{\zeta}(1-\alpha)^2-(1-\zeta)\alpha(1+\alpha)-(1-\zeta)\beta(1+\beta) \geqslant 0. \tag{3.40}$$

Furthermore, we obtain the following:

$$\begin{aligned} \zeta &\leqslant \frac{1}{2} \\ \Rightarrow \zeta+\zeta &\leqslant 1. \end{aligned} \tag{3.41}$$

Moreover, the inequality:

$$\zeta \leqslant 1-\zeta. \tag{3.42}$$

When combining (3.41) with (3.42), it implies the following:

$$\frac{1-\zeta}{\zeta}(1-\alpha)^2-(1-\zeta)\alpha(1+\alpha)-\zeta\beta(1+\beta) \geqslant 0. \tag{3.43}$$

This leads to the conclusion that

$$\rho-\mu \geqslant \left[\frac{(1-\zeta)}{\zeta}(1-\alpha)^2-\alpha(1+\alpha)+\zeta\alpha(1+\alpha)-\zeta\beta(1+\beta)\right] \geqslant 0. \tag{3.44}$$

By expressing (3.34), we can rewrite it as follows:

$$\Psi_{k+1}-\Psi_k \leqslant -(\rho-\mu)\|u_{k+1}-u_k\|^2 \leqslant 0. \tag{3.45}$$

The above analysis indicates that the sequence $\{\Psi_k\}$ is nonincreasing. Let us consider that

$$\epsilon = [(1-\zeta)\alpha+\zeta\beta] \leqslant \max\{\alpha, \beta\} < 1.$$

This expression transforms into

$$\Psi_k = \|u_k-u^*\|^2-\epsilon\|u_{k-1}-u^*\|^2+\mu\|u_k-u_{k-1}\|^2.$$

From Ψ_{k+1}, we have

$$\begin{aligned} \Psi_{k+1} &= \|u_{k+1}-u^*\|^2-\epsilon\|u_k-u^*\|^2+\mu\|u_{k+1}-u_k\|^2 \\ &\geqslant -\epsilon\|u_k-u^*\|^2. \end{aligned} \tag{3.46}$$

Similarly by definition of Ψ_k, we obtain

$$\begin{aligned}\Psi_k &= \|u_k - u^*\|^2 - \epsilon\|u_{k-1} - u^*\|^2 + \mu\|u_k - u_{k-1}\|^2 \\ &\geqslant \|u_k - u^*\|^2 - \epsilon\|u_{k-1} - u^*\|^2.\end{aligned} \tag{3.47}$$

Inequality (3.47) suggests that

$$\begin{aligned}\|u_k - u^*\|^2 &\leqslant \Psi_k + \epsilon\|u_{k-1} - u^*\|^2 \\ &\leqslant \Psi_1 + \epsilon\|u_{k-1} - u^*\|^2 \\ &\leqslant \cdots \leqslant \Psi_1(\epsilon^{k-1} + \cdots + 1) + \epsilon^k\|u_0 - u^*\|^2 \\ &\leqslant \frac{\Psi_1}{1-\epsilon} + \epsilon^k\|u_0 - u^*\|^2.\end{aligned} \tag{3.48}$$

Combining (3.46) and (3.48), we obtain

$$\begin{aligned}-\Psi_{k+1} &\leqslant \epsilon\|u_k - u^*\|^2 \\ &\leqslant \epsilon\frac{\Psi_1}{1-\epsilon} + \epsilon^{k+1}\|u_0 - u^*\|^2.\end{aligned} \tag{3.49}$$

It follows from expressions (3.34) and (3.49) such that

$$\begin{aligned}(\rho - \mu)\sum_{k=1}^{n}\|u_{k+1} - u_k\|^2 &\leqslant \Psi_1 - \Psi_{n+1} \\ &\leqslant \Psi_1 + \epsilon\frac{\Psi_1}{1-\epsilon} + \epsilon^{n+1}\|u_0 - u^*\|^2 \\ &\leqslant \frac{\Psi_1}{1-\epsilon} + \|u_0 - u^*\|^2.\end{aligned} \tag{3.50}$$

By letting $(n \to +\infty)$ in the above expression implies that

$$\sum_{k=1}^{+\infty}\|u_{k+1} - u_k\|^2 < +\infty. \tag{3.51}$$

This follows that

$$\lim_{k\to+\infty}\|u_{k+1} - u_k\| = 0. \tag{3.52}$$

From expressions (3.31), we have

$$\|u_{k+1} - s_k\|^2 = \|u_{k+1} - u_k\|^2 + \alpha^2\|u_k - u_{k-1}\|^2 - 2\alpha\langle u_{k+1} - u_k, u_k - u_{k-1}\rangle \tag{3.53}$$

From expressions (3.52) and (3.53), we obtain

$$\|u_{k+1} - s_k\| \to 0 \text{ as } k \to +\infty. \tag{3.54}$$

Inequality (3.33) with Lemma 2.3 and $\sum_{k=1}^{\infty}\|u_{k+1} - u_k\| < +\infty$ imply that

$$\lim_{k\to+\infty}\|u_k - u^*\|^2 = l, \text{ for some finite } l > 0. \tag{3.55}$$

From expressions (3.28), (3.52), and (3.55), we obtain

$$\lim_{k\to+\infty}\|w_k - u^*\|^2 = l. \tag{3.56}$$

Combining expression (3.26) and (3.54), we have

$$\lim_{k\to+\infty}(q_k - s_k) = \frac{1}{\zeta}\lim_{k\to+\infty}(u_{k+1} - s_k) = 0. \tag{3.57}$$

$$\lim_{k\to+\infty}(s_k - u_k) = \alpha\lim_{k\to+\infty}(u_k - u_{k-1}) = 0. \tag{3.58}$$

$$\lim_{k\to+\infty}(w_k - u_k) = \beta\lim_{k\to+\infty}(u_k - u_{k-1}) = 0. \tag{3.59}$$

Combining expressions (3.58) and (3.59), we have

$$\lim_{k\to+\infty}(s_k - w_k) = 0. \tag{3.60}$$

It follows from (3.57) and (3.60) that

$$\lim_{k\to+\infty}(q_k - w_k) = 0. \tag{3.61}$$

As a result of expressions (3.20) and (3.28), we have

$$\left(1-\mu^2\frac{\gamma_k^2}{\gamma_{k+1}^2}\right)\|w_k - v_k\|^2 \leqslant \|w_k - u^*\|^2 - \|q_k - u^*\|^2. \tag{3.62}$$

$$\begin{aligned}&\leqslant (1+\beta_k)\|u_k - u^*\|^2 - \beta_k\|u_{k-1} - u^*\|^2 + 2\beta_k\|u_k - u_{k-1}\|^2 - \|q_k - u^*\|^2\\ &\leqslant \|u_k - u^*\|^2 - \|q_k - u^*\|^2 + \beta_k\left(\|u_k - u^*\|^2 - \|u_{k-1} - u^*\|^2\right) + 2\beta_k\|u_k - u_{k-1}\|^2.\end{aligned} \tag{3.63}$$

Taking the limit as $k \to +\infty$ in the expression (3.63), we obtain

$$\lim_{k\to+\infty}\|w_k - v_k\| = 0. \tag{3.64}$$

It follows that

$$\|q_k - v_k\| = \|v_k + \gamma_k[\mathcal{P}(w_k) - \mathcal{P}(v_k)] - v_k\| \leqslant \gamma_k L\|w_k - v_k\|.$$

The above expression implies that

$$\lim_{k\to\infty}\|q_k - v_k\| = 0. \tag{3.65}$$

Thus, the expressions (3.58), (3.59), and (3.64) give that

$$\lim_{k\to+\infty}\|q_k - u^*\| = \lim_{k\to+\infty}\|s_k - u^*\| = \lim_{k\to+\infty}\|v_k - u^*\| = l. \tag{3.66}$$

According to the above discussion, the sequences $\{u_k\}$, $\{v_k\}$, $\{w_k\}$, $\{s_k\}$, and $\{q_k\}$ are bounded, and for each $u^* \in \Omega$, exists the $\lim_{k\to+\infty}\|u_k - u^*\|^2$, $\lim_{k\to+\infty}\|v_k - u^*\|^2$, $\lim_{k\to+\infty}\|w_k - u^*\|^2$, $\lim_{k\to+\infty}\|s_k - u^*\|^2$, $\lim_{k\to+\infty}\|q_k - u^*\|^2$. Following that, we will show that the sequence $\{u_k\}$ weakly converges to u^*. As a result, all sequences $\{u_k\}$, $\{w_k\}$ and $\{v_k\}$ are bounded. We now demonstrate that each sequential weak cluster point in the sequence $\{u_k\}$ is in the solution set Ω. Consider that $\hat{u}$ is a weak cluster point of $\{u_k\}$, which means that there is a subsequence $\{u_{k_l}\}$ of $\{u_k\}$ that is weakly convergent to $\hat{u} \in \mathcal{D}$. Also, subsequence $\{v_{k_l}\}$ is weakly convergent to $\hat{u}$. Now let us prove that $\hat{u} \in \Omega$. It is proven before that the sequences $\{w_k\}$, $\{s_k\}$ and $\{v_k\}$ are also bounded sequences. Due to reflexivity of a Hilbert space $\mathcal{E}$ and the boundedness of a sequence $\{u_k\}$ guarantees that there exists a subsequence $\{u_{k_l}\}$ such that $\{u_{k_l}\} \rightharpoonup \hat{u} \in \mathcal{E}$ as $l \to +\infty$. Next, we need to show that $\hat{u} \in \Omega$. By value of v_k, we have

$$v_{k_l} = P_{\mathcal{D}}\left[w_{k_l} - \gamma_{k_l}\mathcal{P}(w_{k_l})\right]$$

that is equivalent can be written as follows:

$$\langle w_{k_l} - \gamma_{k_l}\mathcal{P}(w_{k_l}) - v_{k_l}, v - v_{k_l}\rangle \leqslant 0, \ \forall v \in \mathcal{D}. \tag{3.67}$$

The above inequality further implies that

$$\langle w_{k_l} - v_{k_l}, v - v_{k_l}\rangle \leqslant \gamma_{k_l}\langle \mathcal{P}(w_{k_l}), v - v_{k_l}\rangle, \ \forall v \in \mathcal{D}. \tag{3.68}$$

Thus, we obtain

$$\frac{1}{\gamma_{k_l}}\langle w_{k_l} - v_{k_l}, v - v_{k_l}\rangle + \langle \mathcal{P}(w_{k_l}), v_{k_l} - w_{k_l}\rangle \leqslant \langle \mathcal{P}(w_{k_l}), v - w_{k_l}\rangle, \ \forall v \in \mathcal{D}. \tag{3.69}$$

Due to the boundedness of the sequence $\{w_{k_l}\}$, it implies that $\{\mathcal{P}(w_{k_l})\}$ is also a bounded sequence. By the use of $\lim_{l\to+\infty} \|w_{k_l} - v_{k_l}\| = 0$ and $l \to +\infty$ in expression (3.69), we obtain

$$\liminf_{l\to+\infty}\langle \mathcal{P}(w_{k_l}), v - w_{k_l}\rangle \geqslant 0, \ \forall v \in \mathcal{D}. \tag{3.70}$$

Moreover, we obtain

$$\langle \mathcal{P}(v_{k_l}), v - v_{k_l}\rangle = \langle \mathcal{P}(v_{k_l}) - \mathcal{P}(w_{k_l}), v - w_{k_l}\rangle + \langle \mathcal{P}(w_{k_l}), v - w_{k_l}\rangle + \langle \mathcal{P}(v_{k_l}), w_{k_l} - v_{k_l}\rangle. \tag{3.71}$$

Since $\lim_{l\to+\infty}\|w_{k_l} - v_{k_l}\| = 0$ and $\mathcal{P}$ is L-Lipschitz continuous on $\mathcal{E}$ such that

$$\lim_{l\to+\infty} \|\mathcal{P}(w_{k_l}) - \mathcal{P}(v_{k_l})\| = 0, \tag{3.72}$$

which together with expressions (3.71) and (3.72), we obtain

$$\liminf_{l\to+\infty}\langle \mathcal{P}(v_{k_l}), v - v_{k_l}\rangle \geqslant 0, \ \forall v \in \mathcal{D}. \tag{3.73}$$

Next, let us take a positive sequence $\{\epsilon_l\}$ that is decreasing and converges to zero. We represent m_l by the smallest positive integer for each ϵ_l such that

$$\langle \mathcal{P}(w_{k_i}), v - w_{k_i}\rangle + \epsilon_l \geqslant 0, \ \forall i \geqslant m_l. \tag{3.74}$$

As $\{\epsilon_l\}$ is decreasing, it is easy to see that the sequence $\{m_l\}$ is increasing.

Case I: Let $\{w_{k_{m_{l_j}}}\}$ be a subsequence of a sequence $\{w_{k_{m_l}}\}$ such that $\mathcal{P}(w_{k_{m_{l_j}}}) = 0$ ($\forall j$). Let $j \to \infty$, we get

$$\langle \mathcal{P}(\hat{u}), v - \hat{u}\rangle = \lim_{j\to\infty}\left\langle \mathcal{P}\left(w_{k_{m_{l_j}}}\right), v - \hat{u}\right\rangle = 0. \tag{3.75}$$

As $\hat{u} \in \mathcal{D}$, we have $\hat{u} \in \Omega$.

Case II: If there exists $N_0 \in \mathbb{N}$ such that for all $k_{m_l} \geqslant N_0$, $\mathcal{P}(w_{k_{m_l}}) \neq 0$. Consider that

$$\Upsilon_{k_{m_l}} = \frac{\mathcal{P}(w_{k_{m_l}})}{\|\mathcal{P}(w_{k_{m_l}})\|^2}, \ \forall k_{m_l} \geqslant N_0. \tag{3.76}$$

Due to the above definition, we have

$$\langle \mathcal{P}(w_{k_{m_l}}), \Upsilon_{k_{m_l}}\rangle = 1, \ \forall k_{m_l} \geqslant N_0. \tag{3.77}$$

Moreover, from expressions (3.74) and (3.77) for all $k_{m_l} \geqslant N_0$, we have

$$\langle \mathcal{P}(w_{k_{m_l}}),\ v + \epsilon_l \Upsilon_{k_{m_l}} - w_{k_{m_l}} \rangle \geqslant 0. \tag{3.78}$$

By the definition of pseudomonotone mapping $\mathcal{P}$ for $k_{m_l} \geqslant N_0$, we have

$$\langle \mathcal{P}(v + \epsilon_l \Upsilon_{k_{m_l}}),\ v + \epsilon_l \Upsilon_{k_{m_l}} - w_{k_{m_l}} \rangle \geqslant 0. \tag{3.79}$$

That is, for all $k_{m_l} \geqslant N_0$, we obtain

$$\langle \mathcal{P}(v), v - w_{k_{m_l}} \rangle \geqslant \langle \mathcal{P}(v) - \mathcal{P}(v + \epsilon_l \Upsilon_{k_{m_l}}),\ v + \epsilon_l \Upsilon_{k_{m_l}} - w_{k_{m_l}} \rangle - \epsilon_l \langle \mathcal{P}(v), \Upsilon_{k_{m_l}} \rangle. \tag{3.80}$$

Due to $\{w_{k_l}\}$ weakly converges to $\hat{u} \in \mathcal{D}$ through $\mathcal{P}$ is sequentially weakly continuous on the set $\mathcal{D}$, we get $\{\mathcal{P}(w_{k_l})\}$ weakly converges to $\mathcal{P}(\hat{u})$. Consider that $\mathcal{P}(\hat{u}) \neq 0$, we have

$$\|\mathcal{P}(\hat{u})\| \leqslant \liminf_{l\to\infty} \|\mathcal{P}(w_{k_l})\|. \tag{3.81}$$

Since $\{w_{k_{m_l}}\} \subset \{w_{k_l}\}$ and $\lim_{l\to\infty} \epsilon_l = 0$, we have

$$0 \leqslant \lim_{l\to\infty} \|\epsilon_l \Upsilon_{k_{m_l}}\| = \lim_{l\to\infty} \frac{\epsilon_l}{\|\mathcal{P}(w_{k_{m_l}})\|} \leqslant \frac{0}{\|\mathcal{P}(\hat{u})\|} = 0. \tag{3.82}$$

For letting $l \to +\infty$ in the expression (3.80), we obtain

$$\langle \mathcal{P}(v), v - \hat{u} \rangle \geqslant 0,\ \forall v \in \mathcal{D}. \tag{3.83}$$

This demonstrates that $\hat{u} \in \Omega$. Thus, Lemma 2.4 assures that $\{w_k\}$, $\{u_k\}$ and $\{v_k\}$ converge weakly toj u^* as $k \to +\infty$.

□

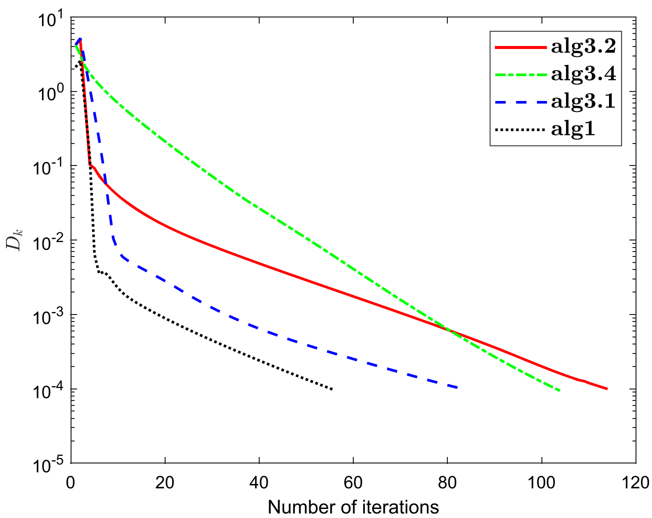

FIGURE 1 The numerical results offered a comparison between our proposed algorithms and existing ones, with a specific focus on the number of iterations. These comparisons were conducted under the consideration of a spatial dimension of $m = 10$ and utilizing Example 4.1. [Colour figure can be viewed at wileyonlinelibrary.com]

4 | NUMERICAL ILLUSTRATIONS

In this section, we conducted a series of numerical experiments to showcase the efficacy of the proposed methodologies. These experiments serve dual purposes: Firstly, they offer valuable insights into the process of choosing optimal control parameters, and secondly, they illustrate the superior performance of our methods when compared to those previously documented in the literature. It is crucial to emphasize that, throughout this section, the error term is consistently denoted as D_k in all methods and computations under consideration. All MATLAB codes were run on a machine with the following specifications: Intel(R) Core(TM) i5-6200 Processor CPU @ 2.30GHz 2.40GHz, and 8.00 GB RAM.

Example 4.1. The first problem is HP hard problem, originally introduced in Harker and Pang [33]. The problem involves a mapping $\mathcal{P}$ that operates from $\mathbb{R}^m$ to $\mathbb{R}^m$. This mapping is defined as follows:

$$\mathcal{P}(u) = Mu + q.$$

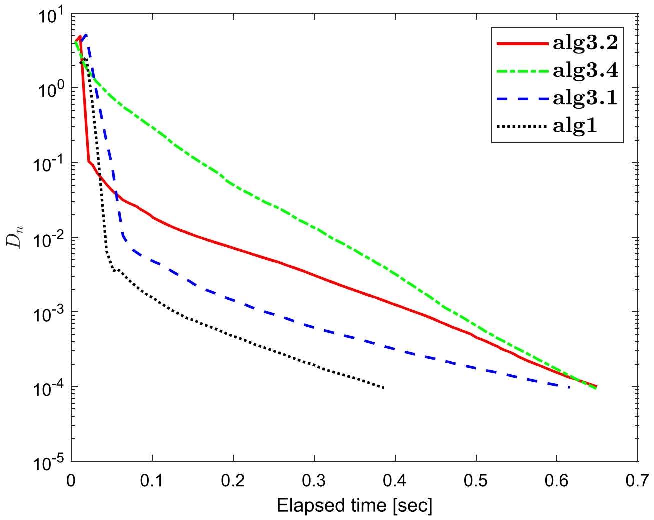

FIGURE 2 The numerical results offered a comparison between our proposed algorithms and existing ones, with a specific focus on the execution time. These comparisons were conducted under the consideration of a spatial dimension of $m = 10$ and utilizing Example 4.1. [Colour figure can be viewed at wileyonlinelibrary.com]

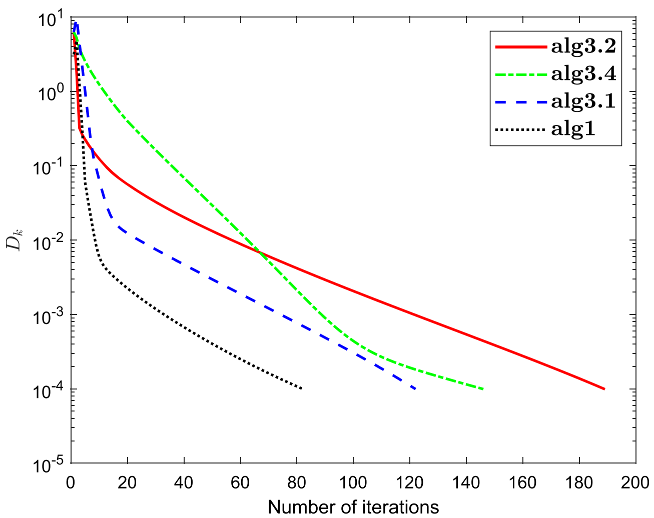

FIGURE 3 The numerical results offered a comparison between our proposed algorithms and existing ones, with a specific focus on the number of iterations. These comparisons were conducted under the consideration of a spatial dimension of $m = 20$ and utilizing Example 4.1. [Colour figure can be viewed at wileyonlinelibrary.com]

Here, the matrix M is defined as follows:

$$M = NN^T + B + D.$$

In the above expressions, q is an element of $\mathbb{R}^m$. To clarify the components involved: N is a random matrix, where $N = \text{rand}(m)$. B is a skew-symmetric matrix, given by $B = 0.5K - 0.5K^T$, where $K = \text{rand}(m)$. D is a diagonal matrix represented as $D = \text{diag}(\text{rand}(m, 1))$. Now, let us define the feasible set $\mathcal{D}$ as follows:

$$\mathcal{D} = \{u \in \mathbb{R}^m : Qu \leqslant b\}.$$

In this definition: Q is a random matrix with dimensions $100 \times m$, that is, $Q = \text{rand}(100, m)$. Moreover, b is a random vector with dimensions 100×1, that is, $b = \text{rand}(100, 1)$. It is worth noting that the mapping $\mathcal{P}$ is both Lipschitz continuous and monotone. The primary aim of the initial experiment is to assess the suggested algorithm under the assumption of knowing the Lipschitz constants. It is important to highlight that, although we have access to all the

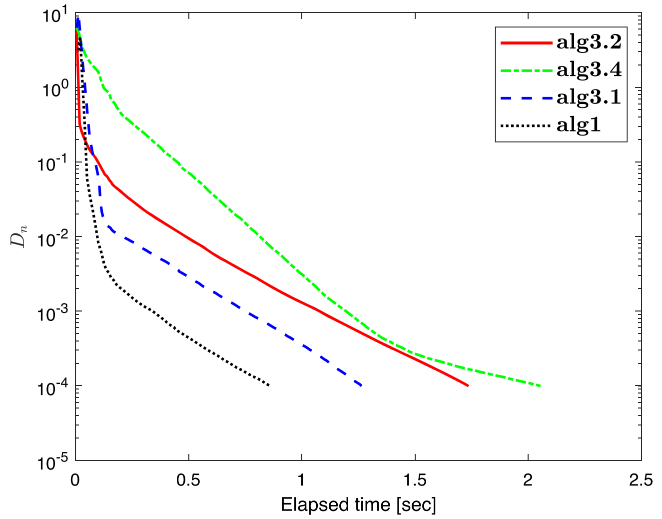

FIGURE 4 The numerical results offered a comparison between our proposed algorithms and existing ones, with a specific focus on the execution time. These comparisons were conducted under the consideration of a spatial dimension of $m = 20$ and utilizing Example 4.1. [Colour figure can be viewed at wileyonlinelibrary.com]

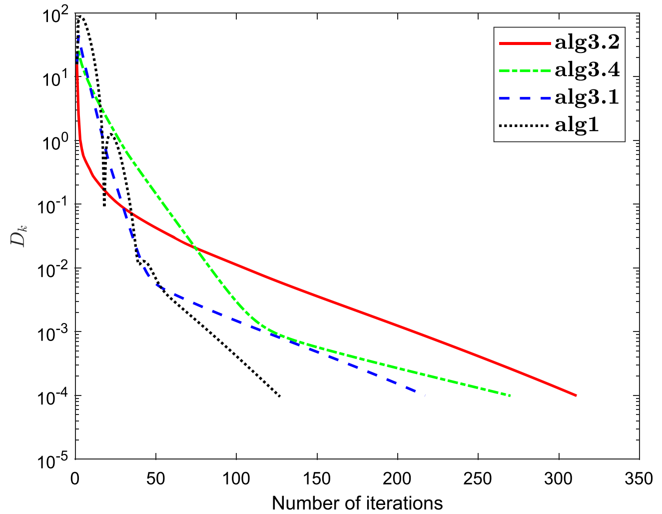

FIGURE 5 The numerical results offered a comparison between our proposed algorithms and existing ones, with a specific focus on the number of iterations. These comparisons were conducted under the consideration of a spatial dimension of $m = 30$ and utilizing Example 4.1. [Colour figure can be viewed at wileyonlinelibrary.com]

proposed algorithms, we specifically employed Algorithm 2 for comparative purposes in this specific scenario. This choice was made due to the variable step-size rule in Algorithm 2, which changes with each iteration, facilitating to fin an optimal parameter. The numerical results are presented in Figures 1 to 8 and Tables 1 and 2. A consistent pattern is evident across all these cases: Our proposed algorithms consistently surpass the performance of pre-existing ones. It is crucial to underscore that the chosen spatial dimension significantly influences computational performance. However, it is essential to acknowledge that algorithm performance is affected by various factors. As the spatial dimension expands, both the number of iterations and the execution time needed to attain a solution also increase. Aside from the spatial dimension, additional criteria come into consideration and can significantly impact the efficacy of all corresponding algorithms.

For all algorithms, we initialize the vectors as follows: $u_{-1} = (1, 1, \cdots, 1, 1_m)^T$, $u_0 = (1, 1, \cdots, 1, 1_m)^T$. The condition for termination is defined as follows: $D_k = \|w_k - v_k\|$. Furthermore, for the purpose of running these algorithms in the MATLAB environment, we have specified the following parameter conditions:

(1) Algorithm 3.2 in Thong et al. [34] (**alg3.2**): $\alpha = 0.65$, $\tau_1 = 0.22$, $\mu = 0.90$, $\beta_k = \frac{1}{(5k+2)}$, $\epsilon_k = \frac{1}{(k+1)^2}$, $f(x) = \frac{x}{2}$.

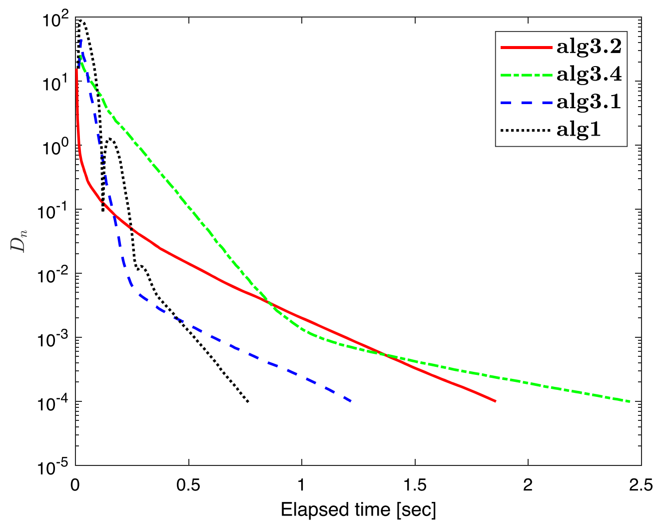

FIGURE 6 The numerical results offered a comparison between our proposed algorithms and existing ones, with a specific focus on the execution time. These comparisons were conducted under the consideration of a spatial dimension of $m = 30$ and utilizing Example 4.1. [Colour figure can be viewed at wileyonlinelibrary.com]

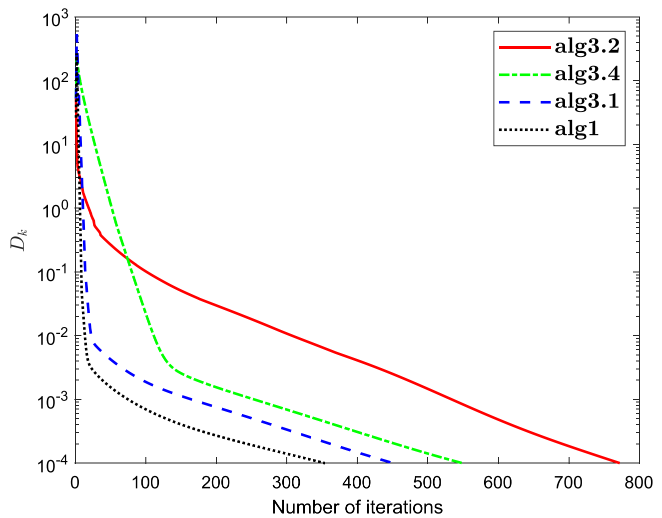

FIGURE 7 The numerical results offered a comparison between our proposed algorithms and existing ones, with a specific focus on the number of iterations. These comparisons were conducted under the consideration of a spatial dimension of $m = 50$ and utilizing Example 4.1. [Colour figure can be viewed at wileyonlinelibrary.com]

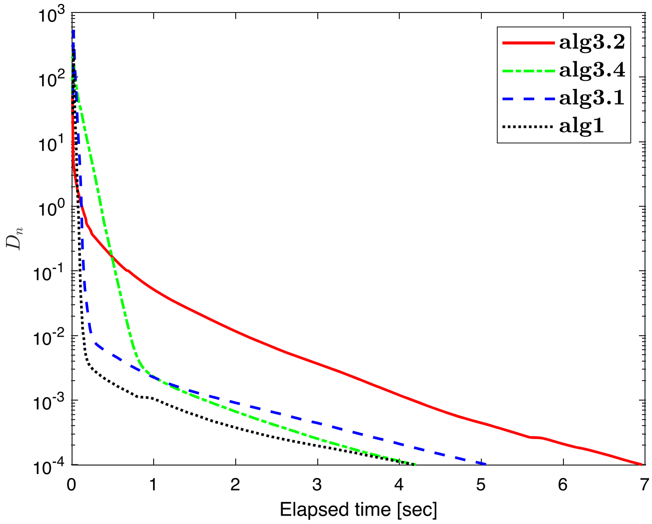

FIGURE 8 The numerical results offered a comparison between our proposed algorithms and existing ones, with a specific focus on the execution time. These comparisons were conducted under the consideration of a spatial dimension of $m = 50$ and utilizing Example 4.1. [Colour figure can be viewed at wileyonlinelibrary.com]

TABLE 1 A detailed numerical results data provided for Figures 1–8.

	Number of iterations			
m	**(alg3.2)**	**(alg3.4)**	**(alg3.1)**	**(alg1)**
10	114	104	84	56
20	189	146	122	82
30	311	270	217	127
50	772	549	451	354

TABLE 2 A detailed numerical results data provided for Figures 1–8.

	Execution time in seconds			
m	**(alg3.2)**	**(alg3.4)**	**(alg3.1)**	**(alg1)**
10	0.6495358000	0.6487179000	0.6158434000	0.3862360000
20	1.7357792000	2.0538113000	1.2634578000	0.8583252000
30	1.8570223000	2.4490376000	1.2166346000	0.7644448000
50	6.9629246000	4.2014935000	5.0742774000	4.1819123000

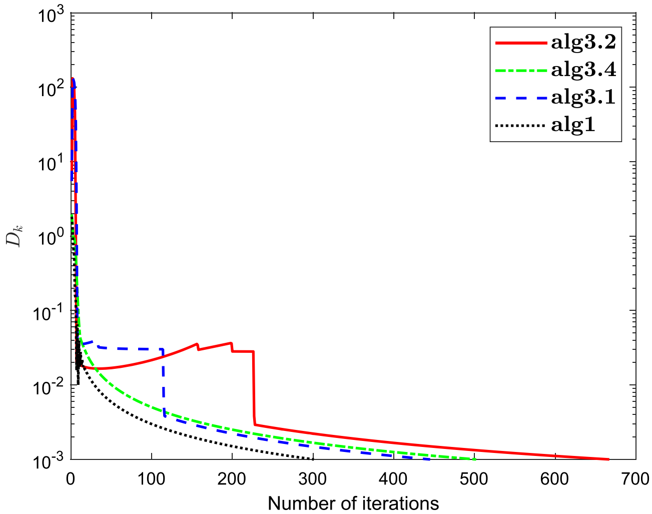

FIGURE 9 The numerical results provide a comparison between the proposed algorithms and existing ones in term of the number of iterations. These comparisons were conducted with the initialization $u_0 = (1, 2, 3, 4)^T$ using Example 4.2. [Colour figure can be viewed at wileyonlinelibrary.com]

(2) Algorithm 3.4 in Anh et al. [35] (**alg3.4**): $\alpha = 0.65, \lambda_0 = 0.22, \mu = 0.90, \tau_k = \frac{1}{(k+1)^2}, \beta_k = \frac{1}{(5k+2)}, \theta_k = \frac{4}{10}(1 - \beta_k)$.
(3) Algorithm 3.1 in Thong et al. [36] (**alg3.1**): $\alpha = 0.65, \lambda = \frac{1}{2L}, \beta_k = \frac{1}{(5k+2)}, \epsilon_k = \frac{1}{(k+1)^2}, f(x) = \frac{x}{2}$.
(4) Algorithm 2 (**alg1**): $\gamma_0 = 0.25, \alpha = 0.55, \beta = 0.454, \mu = 0.90, \zeta = 0.328$.

Example 4.2. Let us consider a mapping $\mathcal{P} : \mathbb{R}^4 \to \mathbb{R}^4$ is defined as follows:

$$\mathcal{P}(u) = \begin{pmatrix} u_1 + u_2 + u_3 + u_4 - 4u_2u_3u_4 \\ u_1 + u_2 + u_3 + u_4 - 4u_1u_3u_4 \\ u_1 + u_2 + u_3 + u_4 - 4u_1u_2u_4 \\ u_1 + u_2 + u_3 + u_4 - 4u_1u_2u_3 \end{pmatrix}.$$

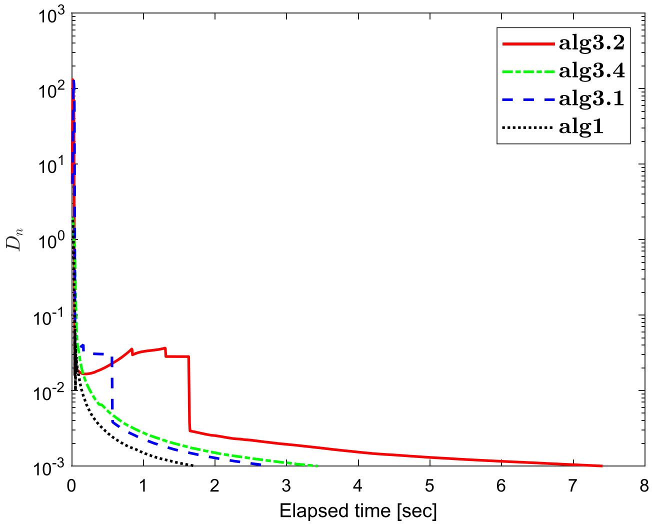

FIGURE 10 The numerical results provide a comparison between the proposed algorithms and existing ones in term of execution time. These comparisons were conducted with the initialization $u_0 = (1, 2, 3, 4)^T$ using Example 4.2. [Colour figure can be viewed at wileyonlinelibrary.com]

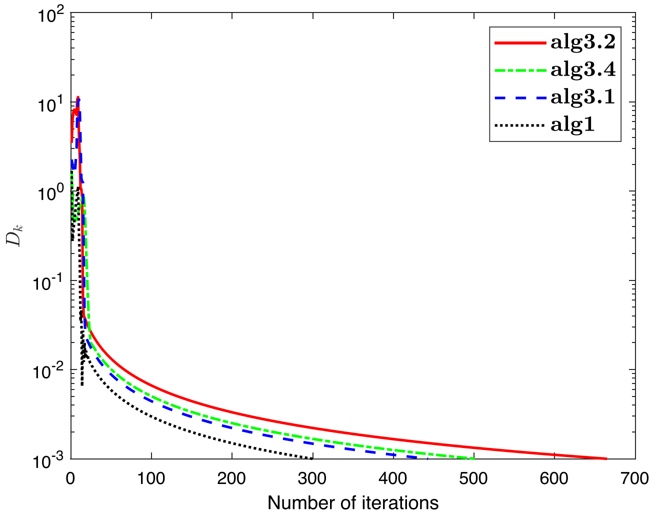

FIGURE 11 The numerical results provide a comparison between the proposed algorithms and existing ones in term of the number of iterations. These comparisons were conducted with the initialization $u_0 = (1, 2, 1, 2)^T$ using Example 4.2. [Colour figure can be viewed at wileyonlinelibrary.com]

Moreover, we define the feasible set $\mathcal{D}$ as follows:

$$\mathcal{D} = \{u \in \mathbb{R}^4 : 1 \leqslant u_i \leqslant 5, i = 1, 2, 3, 4\}.$$

It is evident that the mapping $\mathcal{P}$ does not exhibit monotonic behavior within the set $\mathcal{D}$. It can be proven that $\mathcal{P}$ is pseudomonotone when examined within the bounds of $\mathcal{D}$ using a Monte Carlo technique as discussed in Hu and Wang [37]. Furthermore, it is essential to note that the presented problem has a unique solution $u^* = (5, 5, 5, 5)^T$ and $u_{-1} = (1, 1, 1, 1)^T$. The termination condition is defined as $D_k = \|w_k - v_k\|$. Numerical results are depicted in Figures 9–14 and Tables 3 and 4. A consistent pattern emerges across all these cases: our proposed algorithms consistently surpass their predecessors. It is crucial to highlight that the initial selection of the starting point does not significantly impact computational performance. However, it is essential to acknowledge that algorithm performance is influenced by other parameters as well. The selection of initial points does not significantly affect the number of iterations, although it does lead to variations in the time required to reach a solution. Factors beyond spatial dimensions also play a role and can significantly impact the efficiency of all associated algorithms. Therefore, it is

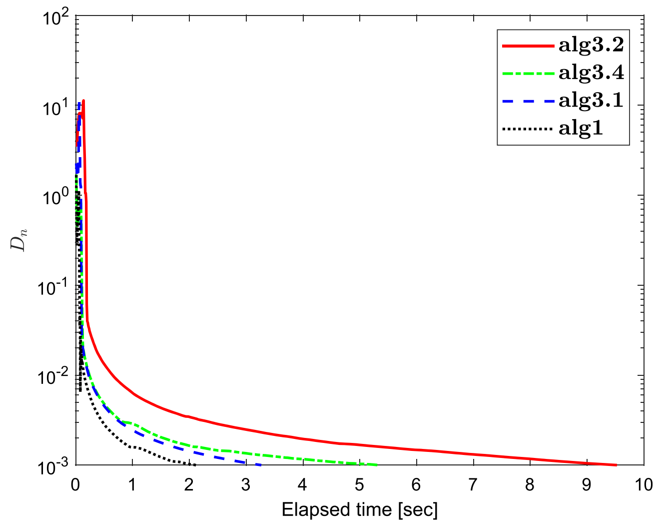

FIGURE 12 The numerical results provide a comparison between the proposed algorithms and existing ones in term of execution time. These comparisons were conducted with the initialization $u_0 = (1, 2, 1, 2)^T$ using Example 4.2. [Colour figure can be viewed at wileyonlinelibrary.com]

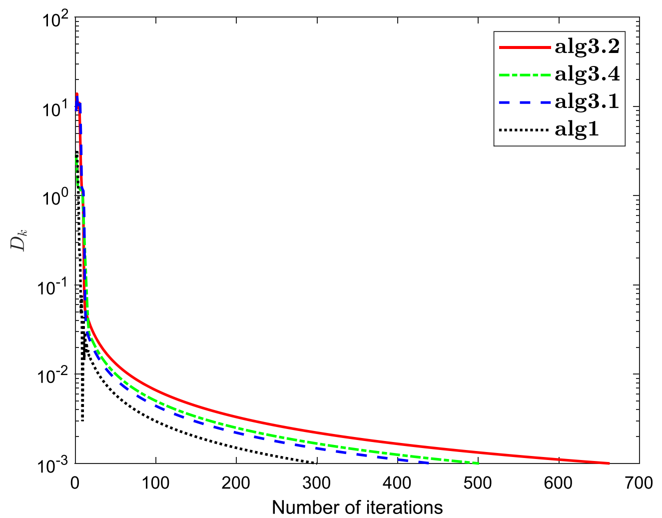

FIGURE 13 The numerical results provide a comparison between the proposed algorithms and existing ones in term of the number of iterations. These comparisons were conducted with the initialization $u_0 = (3, 4, 3, 1)^T$ using Example 4.2. [Colour figure can be viewed at wileyonlinelibrary.com]

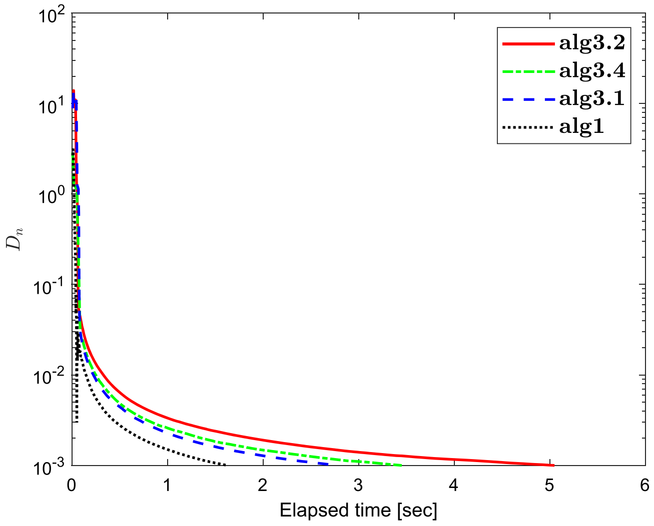

FIGURE 14 The numerical results provide a comparison between the proposed algorithms and existing ones in term of execution time. These comparisons were conducted with the initialization $u_0 = (3, 4, 3, 1)^T$ using Example 4.2. [Colour figure can be viewed at wileyonlinelibrary.com]

TABLE 3 The numerical results data corresponding to Figures 9 through 14 are presented.

	Number of iterations			
u_0	(alg3.2)	(alg3.4)	(alg3.1)	(alg1)
$(1, 2, 3, 4)^T$	667	502	445	301
$(1, 2, 1, 2)^T$	665	501	443	300
$(3, 4, 3, 1)^T$	663	501	442	300

TABLE 4 The numerical results data corresponding to Figures 9 through 14 are presented.

	Execution time in seconds			
u_0	(alg3.2)	(alg3.4)	(alg3.1)	(alg1)
$(1, 2, 3, 4)^T$	7.40983310000000	3.43298360000000	2.78796480000000	1.71841050000000
$(1, 2, 1, 2)^T$	9.52541980000000	5.32200410000000	3.26604860000000	2.11793870000000
$(3, 4, 3, 1)^T$	5.04958530000000	3.45173320000000	2.77985460000000	1.62941670000000

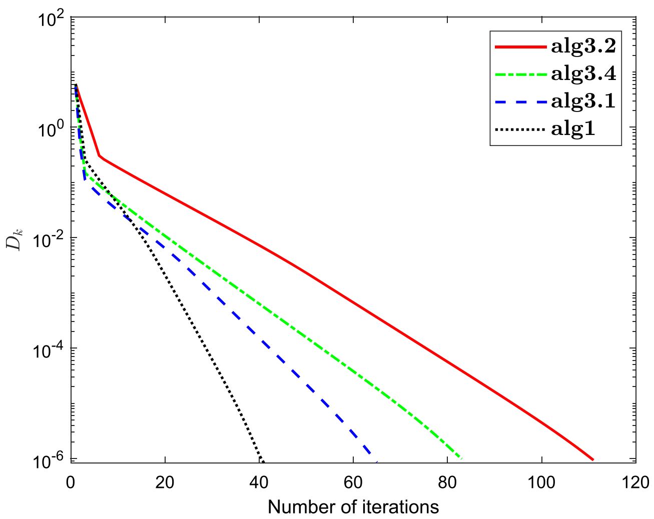

FIGURE 15 The numerical results present a comparison between our proposed algorithms and existing ones, with a focus on the number of iterations. These comparisons were conducted using the initialization $u_0 = 2(t + t^3)$ and referring to Example 4.3. [Colour figure can be viewed at wileyonlinelibrary.com]

crucial to emphasize that we diligently sought to identify optimal parameters for each method, particularly within the context of Example 4.2. The objective of the second experiment was to compare the proposed method under different starting points. For all algorithms, we initialize the parameters as follows:

(1) Algorithm 3.2 in Thong et al. [34] (**alg3.2**): $\alpha = 0.72, \tau_1 = 0.12, \mu = 0.934, \beta_k = \frac{1}{(3k+2)}, \epsilon_k = \frac{1}{(k+1)^2}, f(x) = \frac{x}{2}$.
(2) Algorithm 3.4 in Anh et al. [35] (**alg3.4**): $\alpha = 0.72, \lambda_0 = 0.12, \mu = 0.934, \tau_k = \frac{1}{(k+1)^2}, \beta_k = \frac{1}{(3k+2)}, \theta_k = \frac{5}{10}(1-\beta_k)$.
(3) Algorithm 3.1 in Thong et al. [36] (**alg3.1**): $\alpha = 0.72, \lambda = \frac{1}{2L}, \beta_k = \frac{1}{(3k+2)}, \epsilon_k = \frac{1}{(k+1)^2}, f(x) = \frac{x}{2}$.
(4) Algorithm 2 (**alg1**): $\gamma_0 = 0.32, \alpha = 0.25, \beta = 0.85, \mu = 0.934, \zeta = 0.245, D_k = \|w_k - v_k\|$.

Example 4.3. Consider the Hilbert space $\mathcal{E}$, denoted as $L^2([0,1])$, along with its inner product defined by the following:

$$\langle u, v\rangle = \int_0^1 u(t)v(t)\,dt, \ \forall u, v \in \mathcal{E}.$$

This inner product induces a norm on $\mathcal{E}$, denoted as $\|u\|$, that is given by the following: $\|u\| = \sqrt{\int_0^1 |u(t)|^2\,dt}$. Now, take a unit ball $\mathcal{D}$ within $L^2([0,1])$ as follows: $\mathcal{D} := \{u \in L^2([0,1]) : \|u\| \leqslant 1\}$. Within this framework, we introduce

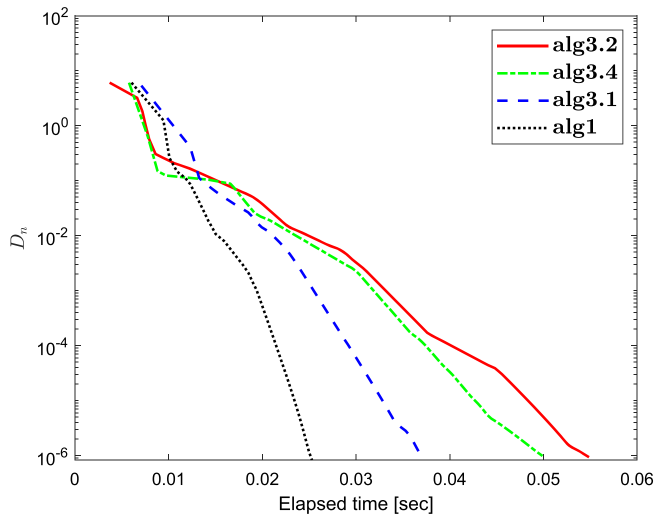

FIGURE 16 The numerical results present a comparison between our proposed algorithms and existing ones, with a focus on the execution time. These comparisons were conducted using the initialization $u_0 = 2(t + t^3)$ and referring to Example 4.3. [Colour figure can be viewed at wileyonlinelibrary.com]

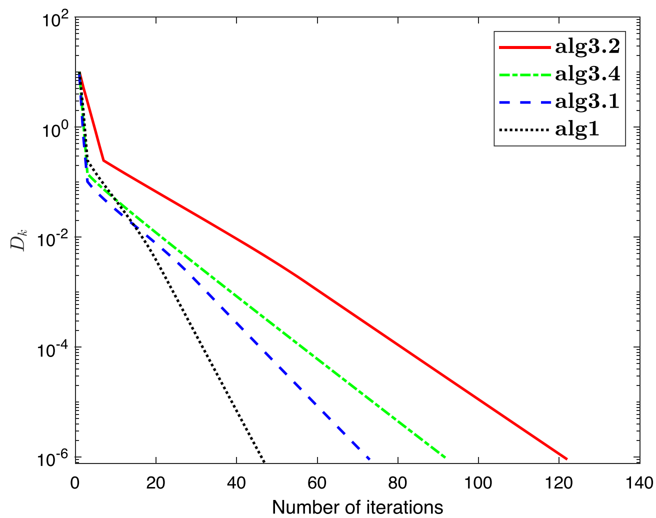

FIGURE 17 The numerical results present a comparison between our proposed algorithms and existing ones, with a focus on the number of iterations. These comparisons were conducted using the initialization $u_0 = 5t^3 \exp(t^3)$ and referring to Example 4.3. [Colour figure can be viewed at wileyonlinelibrary.com]

a mapping $\mathcal{P}$ operating from $\mathcal{D}$ to $\mathcal{E}$, defined as follows:

$$\mathcal{P}(u)(t) = \int_0^1 (u(t) - H(t,s)f(u(s)))\,ds + g(t).$$

Here, the functions $H(t,s)$, $f(u)$, and $g(t)$ are expressed as follows:

$$H(t,s) = \frac{2tse^{(t+s)}}{e\sqrt{e^2-1}},\ \ f(u) = \cos u,\ \ g(t) = \frac{2te^t}{e\sqrt{e^2-1}}.$$

The termination condition is defined as: $D_k = \|w_k - v_k\|$, and $u_{-1} = t$. The numerical results are depicted in Figures 15 through 20, as well as Tables 5 and 6. Additionally, specific parameter conditions have been established for the execution of these algorithms in the MATLAB environment.

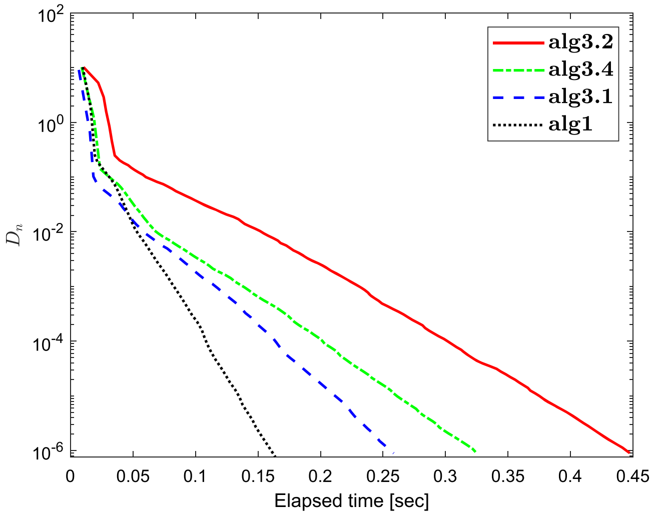

FIGURE 18 The numerical results present a comparison between our proposed algorithms and existing ones, with a focus on the execution time. These comparisons were conducted using the initialization $u_0 = 5t^3\exp(t^3)$ and referring to Example 4.3. [Colour figure can be viewed at wileyonlinelibrary.com]

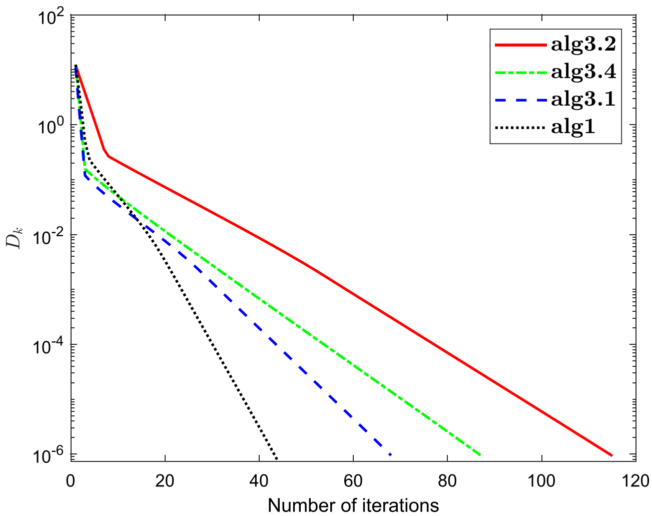

FIGURE 19 The numerical results present a comparison between our proposed algorithms and existing ones, with a focus on the number of iterations. These comparisons were conducted using the initialization $u_0 = 3\sin^2(t)$ and referring to Example 4.3. [Colour figure can be viewed at wileyonlinelibrary.com]

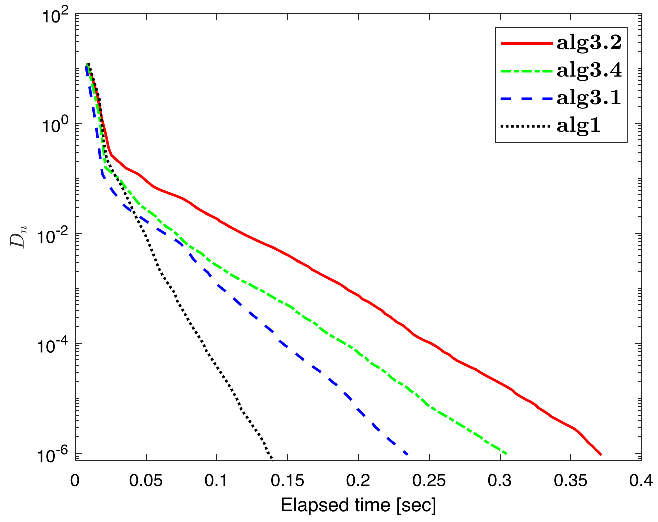

FIGURE 20 The numerical results present a comparison between our proposed algorithms and existing ones, with a focus on the execution time. These comparisons were conducted using the initialization $u_0 = 3\sin^2(t)$ and referring to Example 4.3. [Colour figure can be viewed at wileyonlinelibrary.com]

TABLE 5 The numerical data outcomes for Figures 15 through 20.

	Number of iterations			
u_0	(alg3.2)	(alg3.4)	(alg3.1)	(alg1)
$2(t+t^3)$	111	83	65	41
$5t^3\exp(t^3)$	122	92	73	47
$3\sin^2(t)$	115	87	68	44

TABLE 6 The numerical data outcomes for Figures 15 through 20.

	Execution time in seconds			
u_0	(alg3.2)	(alg3.4)	(alg3.1)	(alg1)
$2(t+t^3)$	0.054863700000	0.04989920000	0.0371664000	0.02528950000
$5t^3\exp(t^3)$	0.447367000000	0.32404780000	0.2584931000	0.16380220000
$3\sin^2(t)$	0.371565000000	0.30556390000	0.2347713000	0.13997550000

(1) Algorithm 3.2 in Thong et al. [34] (**alg3.2**): $\alpha = 0.45, \tau_1 = 0.45, \mu = 0.85, \beta_k = \frac{1}{(4k+2)}, \epsilon_k = \frac{1}{(k+1)^2}, f(x) = \frac{x}{2}$.
(2) Algorithm 3.4 in Anh et al. [35] (**alg3.4**): $\alpha = 0.45, \lambda_0 = 0.45, \mu = 0.85, \tau_k = \frac{1}{(k+1)^2}, \beta_k = \frac{1}{(4k+2)}, \theta_k = \frac{5}{10}(1-\beta_k)$.
(3) Algorithm 3.1 in Thong et al. [36] (**alg3.1**): $\alpha = 0.45, \lambda = \frac{1}{2L}, \beta_k = \frac{1}{(4k+2)}, \epsilon_k = \frac{1}{(k+1)^2}, f(x) = \frac{x}{2}$.
(4) Algorithm 2 (**alg1**): $\gamma_0 = 0.45, \alpha = 0.25, \beta = 0.95, \mu = 0.85, \zeta = 0.328$.

5 | CONCLUSION

To summarize, our study has introduced a new improvement of Tseng's extragradient method, aiming to improve convergence rates while concurrently reducing computational complexity, with a particular emphasis on cost reduction. The proposed methods are specifically designed for solving VIPs in real Hilbert spaces. Initially, we achieved weak convergence results contingent upon the operator satisfying pseudomonotonicity and Lipschitz continuity conditions. It is noteworthy that the step-size rule we propose does not rely on Lipschitz continuity. To showcase the practical utility and real-world applicability of our method, we conducted various numerical examples. This study represents a notable advancement in optimizing methods for solving VIPs.

AUTHOR CONTRIBUTIONS

Nuttapol Pakkaranang: Writing—review and editing; writing—original draft; funding acquisition; validation; methodology; conceptualization; investigation; formal analysis; software; visualization; project administration; data curation; supervision; resources.

ACKNOWLEDGEMENTS

The author would like to thank Professor Dr. Poom Kumam from King Mongkut's University of Technology Thonburi, Thailand for his advice and comments to improve the results of this paper. This work (grant no. RGNS 65-168) was supported by the Office of the Permanent Secretary, Ministry of Higher Education, Science, Research and Innovation (OPS MHESI), Thailand Science Research and Innovation (TSRI), and Phetchabun Rajabhat University.

CONFLICTS OF INTEREST STATEMENT

This work does not have any conflicts of interest.

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How to cite this article: N. Pakkaranang, *Double inertial extragradient algorithms for solving variational inequality problems with convergence analysis*, Math. Meth. Appl. Sci. (2024), 1–28, DOI 10.1002/mma.10147.