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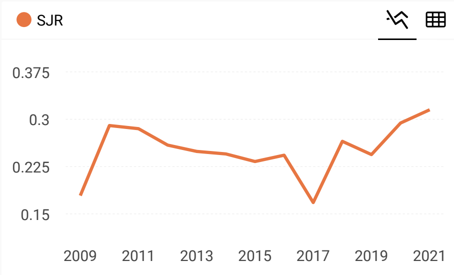

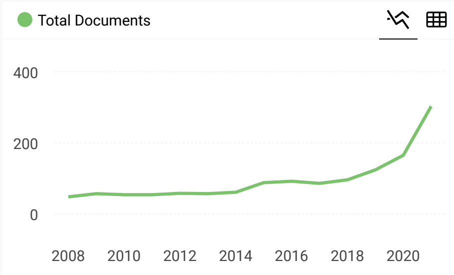

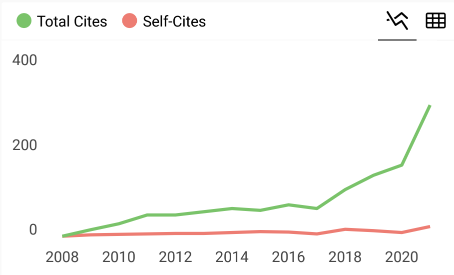

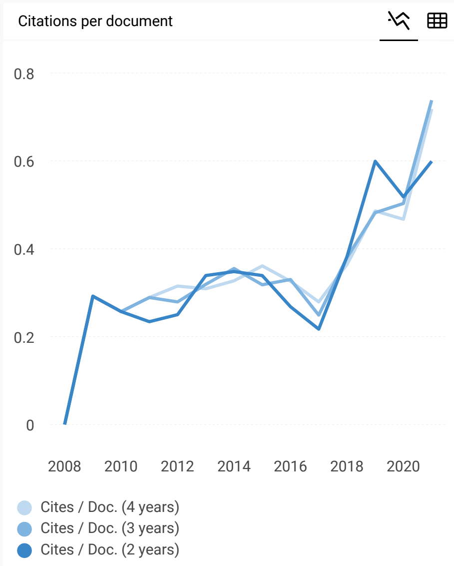

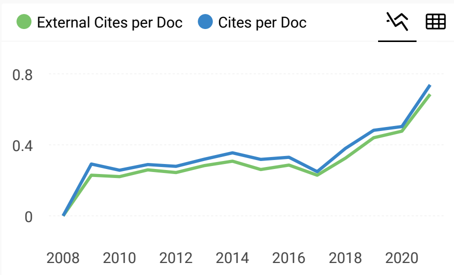

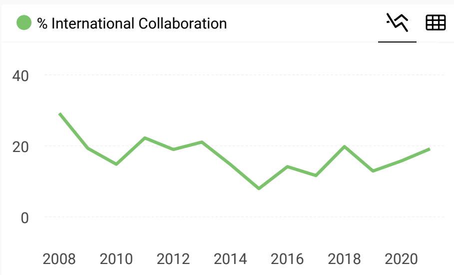

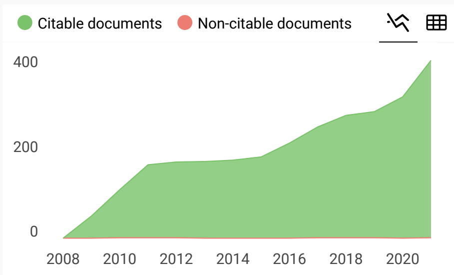

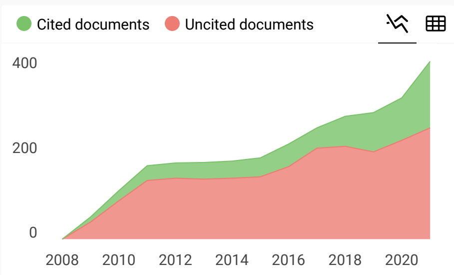

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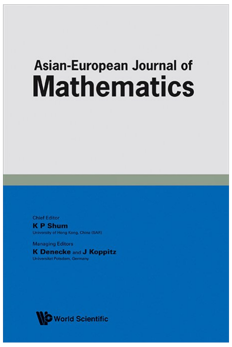

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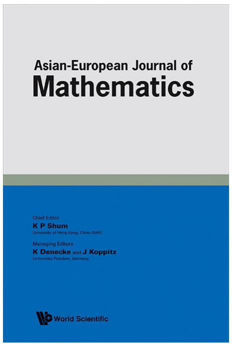

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Strong convergence analysis of modified Mann-type forward–backward scheme for solving quasimonotone variational inequalities

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The paper proposes multiple new extragradient methods for solving a variational inequality problem involving quasimonotone operators in infinite-dimensional real Hilbert spaces. These methods contain variable stepsize rules that are revised at each iteration and are dependent on prior iterations. These algorithms have the benefit of not requiring prior knowledge of the Lipschitz constant or any line-search approach. Simple conditions are used to demonstrate the algorithm's convergence. A collection of simple experiments is presented to show the numerical behavior of the algorithms.

Keywords: Variational inequalities; Tseng's extragradient method; Mann-type iterative scheme; Quasimonotone operator; Lipschitz continuous.

AMS Subject Classification: 47H05, 47H10, 65J15, 47J25, 47J20, 91B50

1. Introduction

The primary objective of this research is to investigate the iterative methods used to figure out the solution to *variational inequality problem* (VIP) [25] problem involving quasimonotone operators in any real Hilbert space. Suppose that Σ is a

†Corresponding author.

real Hilbert space and Δ is a nonempty, closed, and convex subset of Σ. Let an operator $\Gamma : \Sigma \to \Sigma$. The variational inequality problem for Γ on Δ is illustrated in the following manner:

$$\text{Find } \omega^* \in \Delta \quad \text{such that } \langle \Gamma(\omega^*), v - \omega^* \rangle \geq 0, \quad \forall v \in \Delta. \tag{VIP}$$

The mathematical design of the variational inequality problem is a key problem in nonlinear analysis. It is an important mathematical model that develops a lot of fundamental concepts in applied mathematics such as a nonlinear system of equations, optimization conditions for problems with the optimization process, complementarity problems, network equilibrium problems, and finance (see for more details [11, 15–18, 23]). As a consequence, this notion has various applications in the fields of mathematical programming, engineering, transport analysis, network economics, game theory, and computer science. The regularized method and the projection technique are two outstanding and general schemes for finding a solution to variational inequalities. It is also noted that the first approach is most frequently used to deal with the variational inequalities characterized by the class of monotone operators. The regularized sub-problem in this method is strongly monotone, and its unique solution is found to be more convenient than the initial problem. In this study, we discuss the projection methods that are well known for their simpler numerical computing. Many authors have committed themselves to considering not only the theory of existence and stability of solutions but also iterative methods for solving variational inequality problems.

In addition, projection methods are effective in estimating the numerical solution of variational inequalities. Many researchers have provided distinctive variations of projection methods to deal with such problems (see for more details [3, 5, 6, 9, 10, 12–14, 19, 24, 27, 29, 30, 33, 35–37]) and others in [4, 7, 8, 22, 28, 31, 32, 38]. All methods for figuring out the problem (VIP) are based on the computation of a projection on the appropriate set Δ. Korpelevich [19] and Antipin [1] introduced the accompanying extragradient method. Their method takes the following design:

$$\begin{cases} u_1 \in \Delta, \\ v_n = P_\Delta[u_n - \varkappa\Gamma(u_n)], \\ u_{n+1} = P_\Delta[u_n - \varkappa\Gamma(v_n)], \end{cases} \tag{1.1}$$

where $0 < \varkappa < \frac{1}{L}$. In view of the above method, we have used two projections on the underlying set Δ for each iteration. This, of course, can affect the computational effectiveness of the method if the feasible set Δ has a complicated structure. Here, we present some methods which can remove this drawback. The first is the following subgradient extragradient method introduced by Censor *et al.* [9]. This method takes the following form:

$$\begin{cases} u_1 \in \Delta, \\ v_n = P_\Delta[u_n - \varkappa\Gamma(u_n)], \\ u_{n+1} = P_{\Sigma_n}[u_n - \varkappa\Gamma(v_n)], \end{cases} \tag{1.2}$$

where $0 < \varkappa < \frac{1}{L}$ and

$$\Sigma_n = \{z \in \Sigma : \langle u_n - \varkappa\Gamma(u_n) - v_n, z - v_n\rangle \leq 0\}.$$

In this paper, our primary concentrate on Tseng's extragradient method [26] that needs only one projection for each iteration. This method chooses the subsequent manner:

$$\begin{cases} u_1 \in \Delta, \\ v_n = P_\Delta[u_n - \varkappa\Gamma(u_n)], \\ u_{n+1} = v_n + \varkappa[\Gamma(u_n) - \Gamma(v_n)], \end{cases} \tag{1.3}$$

where $0 < \varkappa < \frac{1}{L}$. It is significant to indicate that the above-suggested methods have two serious flaws: a fixed constant step size rule that is conditional on the Lipschitz constant of mapping and develops a weakly convergent iterative sequence. The Lipschitz constant is commonly unknown or challenging to figure out. From a computational point of view, it can be difficult to consider a fixed step size constraint that influences the method's performance and rate of convergence. In addition, the study of a strongly convergent iterative sequence is significant in the situation of an infinite-dimensional Hilbert space.

The primary objective of this study is to set up a new strongly convergent method by applying Mann and Tseng's extragradient-type method, comprising a monotonic and non-monotonic variable step size rule to figure out variational inequalities involving the quasimonotone operator. Furthermore, to show that the iterative sequences set up by all extragradient algorithms strongly converge to a solution. We used the following conditions to examine strong convergence theorems: (Γ1) The solution set for problem (VIP) is denoted by $VI(\Delta, \Gamma)$ which is nonempty; (Γ2) An operator $\Gamma : \Sigma \to \Sigma$ is said to be quasimonotone if

$$\langle\Gamma(u), v - u\rangle > 0 \Rightarrow \langle\Gamma(v), v - u\rangle \geq 0, \quad \forall\, u, v \in \Delta;$$

(Γ3) An operator $\Gamma : \Sigma \to \Sigma$ is said to be *Lipschitz continuous* if there exists a constant $L > 0$ such that

$$\|\Gamma(u) - \Gamma(v)\| \leq L\|u - v\|, \quad \forall\, u, v \in \Delta;$$

(Γ4) An operator $\Gamma : \Sigma \to \Sigma$ is *sequentially weakly continuous* if $\{\Gamma(u_n)\}$ weakly converges to $\Gamma(u)$ for every sequence $\{u_n\}$ weakly converges to u.

The paper is arranged in the following way. In Sec. 2, preliminary results were presented. Section 3 gives all new algorithms and their convergence analysis. Finally, Sec. 4 gives some numerical results to explain the practical efficiency of the proposed methods.

2. Preliminaries

This part comprises a number of significant identities as well as relevant lemmas. For any $u, v \in \Sigma$, we have

$$\|u + v\|^2 = \|u\|^2 + 2\langle u, v\rangle + \|v\|^2.$$

Lemma 2.1 ([2]). *For any $v_1, v_2 \in \Sigma$ and $d \in \mathbb{R}$. Then, the following inequalities hold. (i)* $\|dv_1 + (1-d)v_2\|^2 = d\|v_1\|^2 + (1-d)\|v_2\|^2 - d(1-d)\|v_1 - v_2\|^2$. *(ii)* $\|v_1 + v_2\|^2 \leq \|v_1\|^2 + 2\langle v_2, v_1 + v_2\rangle$.

A metric projection $P_\Delta(v_1)$ of $v_1 \in \Sigma$ *is defined by*

$$P_\Delta(v_1) = \arg\min\{\|v_1 - v_2\| : v_2 \in \Delta\}.$$

Lemma 2.2 ([2]). *Let $P_\Delta : \Sigma \to \Delta$ be a metric projection. Then, the following conditions are satisfied:*

(i) $v_3 = P_\Delta(v_1)$ *if and only if* $\langle v_1 - v_3, v_2 - v_3\rangle \leq 0, \ \forall\, v_2 \in \Delta$;
(ii) $\|v_1 - P_\Delta(v_2)\|^2 + \|P_\Delta(v_2) - v_2\|^2 \leq \|v_1 - v_2\|^2, \ v_1 \in \Delta, v_2 \in \Sigma$;
(iii) $\|v_1 - P_\Delta(v_1)\| \leq \|v_1 - v_2\|, \ v_2 \in \Delta, v_1 \in \Sigma$.

Lemma 2.3 ([34]). *Let $\{e_n\} \subset [0, +\infty)$ be a sequence satisfying the following condition:*

$$e_{n+1} \leq (1 - f_n)e_n + f_n g_n, \quad \forall\, n \in \mathbb{N}.$$

In addition, two sequences $\{f_n\} \subset (0,1)$ and $\{g_n\} \subset \mathbb{R}$ satisfy the following conditions:

$$\lim_{n\to+\infty} f_n = 0, \quad \sum_{n=1}^{+\infty} f_n = +\infty \quad \textit{and} \quad \limsup_{n\to+\infty} g_n \leq 0.$$

Then, $\lim_{n\to+\infty} e_n = 0$.

Lemma 2.4 ([20]). *Let $\{e_n\} \subset \mathbb{R}$ be a sequence and there exists a subsequence $\{n_i\}$ of $\{n\}$ such that*

$$e_{n_i} < e_{n_{i+1}} \quad \forall\, i \in \mathbb{N}.$$

Then, there exists a nondecreasing sequence $m_k \subset \mathbb{N}$ such that $m_k \to +\infty$ as $k \to +\infty$, with

$$e_{m_k} \leq e_{m_{k+1}} \quad \textit{and} \quad e_k \leq e_{m_{k+1}}, \quad \forall\, k \in \mathbb{N}.$$

Indeed, $m_k = \max\{j \leq k : e_j \leq e_{j+1}\}$.

3. Main Results

In this part, we present an iterative scheme for solving quasimonotone variational inequality problems that is based on Tseng's extragradient method [26] and and Mann type scheme [21]. It is important to acknowledge that our approach has a simple framework for obtaining a strong convergence. The first method is given as follows.

Algorithm 1. (Mann-type extragradient method with fixed step size rule)

STEP 0. Let $u_1 \in \Delta$, $0 < \varkappa < \frac{1}{L}$ and $\{\alpha_n\} \subset (a, b) \subset (0, 1 - \vartheta_n)$ and $\{\vartheta_n\} \subset (0, 1)$ satisfies the conditions: $\lim_{n\to+\infty} \vartheta_n = 0$ and $\sum_{n=1}^{+\infty} \vartheta_n = +\infty$.

STEP 1. Compute

$$v_n = P_\Delta(u_n - \varkappa\Gamma(u_n)).$$

If $u_n = v_n$, STOP. Otherwise, go to Step 2.

STEP 2. Compute

$$z_n = v_n + \varkappa[\Gamma(u_n) - \Gamma(v_n)].$$

STEP 3. Compute

$$u_{n+1} = (1 - \alpha_n - \vartheta_n)u_n + \alpha_n z_n.$$

Set $n := n + 1$ and go back to **STEP 1.**

Algorithm 2 . (Mann-type extragradient method with monotonic variable step size rule)

STEP 0. Let $u_1 \in \Delta$, $\varkappa_1 > 0$, $\{\alpha_n\} \subset (a, b) \subset (0, 1 - \vartheta_n)$ and $\{\vartheta_n\} \subset (0, 1)$ satisfies the conditions: $\lim_{n\to+\infty} \vartheta_n = 0$ and $\sum_{n=1}^{+\infty} \vartheta_n = +\infty$.

STEP 1. Compute

$$v_n = P_\Delta(u_n - \varkappa_n\Gamma(u_n)).$$

If $u_n = v_n$, STOP. Otherwise, go to Step 2.

STEP 2. Compute

$$z_n = v_n + \varkappa_n[\Gamma(u_n) - \Gamma(v_n)].$$

STEP 3. Compute

$$u_{n+1} = (1 - \alpha_n - \vartheta_n)u_n + \alpha_n z_n.$$

STEP 4. Compute

$$\varkappa_{n+1} = \begin{cases} \min\left\{\varkappa_n, \dfrac{\chi\|u_n - v_n\|}{\|\Gamma(u_n) - \Gamma(v_n)\|}\right\} & \text{if } \Gamma(u_n) - \Gamma(v_n) \neq 0, \\ \varkappa_n & \text{otherwise.} \end{cases} \tag{3.1}$$

Set $n := n + 1$ and go back to Step 1.

Lemma 3.1. *The sequence* $\{\varkappa_n\}$ *generated by* (3.1) *is decreasing monotonically and converges to* $\varkappa > 0$.

Proof. It is given that Γ is Lipschitz-continuous with constant $L > 0$. Let $\Gamma(u_n) \neq \Gamma(v_n)$ such that

$$\frac{\chi\|u_n - v_n\|}{\|\Gamma(u_n) - \Gamma(v_n)\|} \geq \frac{\chi\|u_n - v_n\|}{L\|u_n - v_n\|} \geq \frac{\chi}{L}. \tag{3.2}$$

The above expression implies that $\lim_{n\to+\infty} \varkappa_n = \varkappa$. □

Algorithm 3. (Mann-type extragradient method with non-monotonic variable step size rule)

STEP 0. Let $u_1 \in \Delta$, $\varkappa_1 > 0$, $\chi \in (0,1)$ and sequence $\{\varphi_n\}$ satisfying $\sum_{n=1}^{+\infty} \varphi_n < +\infty$. Moreover, $\{\alpha_n\} \subset (a,b) \subset (0, 1-\vartheta_n)$ and $\{\vartheta_n\} \subset (0,1)$ satisfies the conditions:

$$\lim_{n\to+\infty} \vartheta_n = 0 \quad \text{and} \quad \sum_{n=1}^{+\infty} \vartheta_n = +\infty.$$

STEP 1. Compute

$$v_n = P_\Delta(u_n - \varkappa_n \Gamma(u_n)).$$

If $u_n = v_n$, STOP. Otherwise, go to Step 2.
STEP 2. Compute

$$z_n = v_n + \varkappa_n[\Gamma(u_n) - \Gamma(v_n)].$$

STEP 3. Compute

$$u_{n+1} = (1 - \alpha_n - \vartheta_n)u_n + \alpha_n z_n.$$

STEP 4. Compute

$$\varkappa_{n+1} = \begin{cases} \min\left\{\varkappa_n + \varphi_n, \dfrac{\chi\|u_n - v_n\|}{\|\Gamma(u_n) - \Gamma(v_n)\|}\right\} & \text{if } \Gamma(u_n) - \Gamma(v_n) \neq 0, \\ \varkappa_n + \varphi_n & \text{otherwise.} \end{cases} \tag{3.3}$$

Set $n := n+1$ and go back to Step 1.

Lemma 3.2. *A sequence* $\{\varkappa_n\}$ *generated by* (3.3) *is convergent to* $\varkappa$ *and satisfying the following inequality:*

$$\min\left\{\frac{\chi}{L}, \varkappa_1\right\} \leq \varkappa_n \leq \varkappa_1 + P, \quad \textit{where } P = \sum_{n=1}^{+\infty} \varphi_n.$$

Proof. It is given that Γ is Lipschitz-continuous with constant $L > 0$. Let $\Gamma(u_n) \neq \Gamma(v_n)$ such that

$$\frac{\chi\|u_n - v_n\|}{\|\Gamma(u_n) - \Gamma(v_n)\|} \geq \tfrac{\chi\|u_n - v_n\|}{L\|u_n - v_n\|} \geq \tfrac{\chi}{L}. \tag{3.4}$$

By using mathematical induction on the definition of $\varkappa_{n+1}$, we have

$$\min\left\{\frac{\chi}{L}, \varkappa_1\right\} \leq \varkappa_n \leq \varkappa_1 + P.$$

Let $[\varkappa_{n+1} - \varkappa_n]^+ = \max\{0, \varkappa_{n+1} - \varkappa_n\}$ and $[\varkappa_{n+1} - \varkappa_n]^- = \max\{0, -(\varkappa_{n+1} - \varkappa_n)\}$. From the definition of $\{\varkappa_n\}$, we have

$$\sum_{n=1}^{+\infty}(\varkappa_{n+1} - \varkappa_n)^+ = \sum_{n=1}^{+\infty}\max\{0, \varkappa_{n+1} - \varkappa_n\} \leq P < +\infty. \tag{3.5}$$

That is, the series $\sum_{n=1}^{+\infty}(\varkappa_{n+1} - \varkappa_n)^+$ is convergent. Next, we need to prove the convergence of $\sum_{n=1}^{+\infty}(\varkappa_{n+1} - \varkappa_n)^-$. Let $\sum_{n=1}^{+\infty}(\varkappa_{n+1} - \varkappa_n)^- = +\infty$. Due to the reason that $\varkappa_{n+1} - \varkappa_n = (\varkappa_{n+1} - \varkappa_n)^+ - (\varkappa_{n+1} - \varkappa_n)^-$. Thus, we have

$$\varkappa_{k+1} - \varkappa_1 = \sum_{n=0}^{k}(\varkappa_{n+1} - \varkappa_n) = \sum_{n=0}^{k}(\varkappa_{n+1} - \varkappa_n)^+ - \sum_{n=0}^{k}(\varkappa_{n+1} - \varkappa_n)^-. \tag{3.6}$$

By allowing $k \to +\infty$ in (3.6), we have $\varkappa_k \to -\infty$ as $k \to +\infty$. This is a contradiction. Due to the convergence of the series $\sum_{n=0}^{k}(\varkappa_{n+1} - \varkappa_n)^+$ and $\sum_{n=0}^{k}(\varkappa_{n+1} - \varkappa_n)^-$ taking $k \to +\infty$ in expression (3.6), we obtain $\lim_{n\to+\infty}\varkappa_n = \varkappa$. This completes the proof of lemma. □

Lemma 3.3. *Suppose that $\Gamma : \Sigma \to \Sigma$ satisfies the conditions (Γ1)–(Γ4) and sequence $\{u_n\}$ generated by Algorithm 1. Then, we have*

$$\|z_n - \omega^*\|^2 \leq \|u_n - \omega^*\|^2 - (1 - \varkappa^2 L^2)\|u_n - v_n\|^2.$$

Proof. Since $\omega^* \in VI(\Delta, \Gamma)$, we have

$$\begin{aligned}
\|z_n - \omega^*\|^2 &= \|v_n + \varkappa[\Gamma(u_n) - \Gamma(v_n)] - \omega^*\|^2 \\
&= \|v_n - \omega^*\|^2 + \varkappa^2\|\Gamma(u_n) - \Gamma(v_n)\|^2 + 2\varkappa\langle v_n - \omega^*, \Gamma(u_n) - \Gamma(v_n)\rangle \\
&= \|v_n + u_n - u_n - \omega^*\|^2 + \varkappa^2\|\Gamma(u_n) - \Gamma(v_n)\|^2 \\
&\quad + 2\varkappa\langle v_n - \omega^*, \Gamma(u_n) - \Gamma(v_n)\rangle \\
&= \|v_n - u_n\|^2 + \|u_n - \omega^*\|^2 + 2\langle v_n - u_n, u_n - \omega^*\rangle \\
&\quad + \varkappa^2\|\Gamma(u_n) - \Gamma(v_n)\|^2 + 2\varkappa\langle v_n - \omega^*, \Gamma(u_n) - \Gamma(v_n)\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \|u_n - \omega^*\|^2 + \|v_n - u_n\|^2 + 2\langle v_n - u_n, v_n - \omega^*\rangle \\
&\quad + 2\langle v_n - u_n, u_n - v_n\rangle + \varkappa^2\|\Gamma(u_n) - \Gamma(v_n)\|^2 \\
&\quad + 2\varkappa\langle v_n - \omega^*, \Gamma(u_n) - \Gamma(v_n)\rangle.
\end{aligned} \tag{3.7}
$$

It provides that $v_n = P_\Delta[u_n - \varkappa\Gamma(u_n)]$ and it implies that

$$\langle u_n - \varkappa\Gamma(u_n) - v_n, v - v_n\rangle \le 0, \quad \forall v \in \Delta. \tag{3.8}$$

Thus, we have

$$\langle u_n - v_n, \omega^* - v_n\rangle \le \varkappa\langle\Gamma(u_n), \omega^* - v_n\rangle. \tag{3.9}$$

Combining expressions (3.7) and (3.9), we obtain

$$
\begin{aligned}
\|z_n - \omega^*\|^2 &\le \|u_n - \omega^*\|^2 + \|v_n - u_n\|^2 + 2\varkappa\langle\Gamma(u_n), \omega^* - v_n\rangle - 2\langle u_n - v_n, \\
&\quad u_n - v_n\rangle + \varkappa^2\|\Gamma(u_n) - \Gamma(v_n)\|^2 - 2\varkappa\langle\Gamma(u_n) - \Gamma(v_n), \omega^* - v_n\rangle \\
&= \|u_n - \omega^*\|^2 - \|u_n - v_n\|^2 + \varkappa^2\|\Gamma(u_n) - \Gamma(v_n)\|^2 \\
&\quad - 2\varkappa\langle\Gamma(v_n), v_n - \omega^*\rangle.
\end{aligned} \tag{3.10}
$$

It is given that ω^* is the solution of the problem (VIP) which implies that

$$\langle\Gamma(\omega^*), v - \omega^*\rangle \ge 0, \quad \forall v \in \Delta.$$

It implies that

$$\langle\Gamma(v), v - \omega^*\rangle \ge 0, \quad \forall v \in \Delta.$$

By substituting $v = v_n \in \Delta$, we have

$$\langle\Gamma(v_n), v_n - \omega^*\rangle \ge 0. \tag{3.11}$$

From expressions (3.10) and (3.11), we obtain

$$
\begin{aligned}
\|z_n - \omega^*\|^2 &\le \|u_n - \omega^*\|^2 - \|u_n - v_n\|^2 + \varkappa^2 L^2\|u_n - v_n\|^2 \\
&= \|u_n - \omega^*\|^2 - (1 - \varkappa^2 L^2)\|u_n - v_n\|^2.
\end{aligned} \tag{3.12}
$$

□

Lemma 3.4. *Assume that $\Gamma : \Sigma \to \Sigma$ satisfies the conditions (Γ1)–(Γ4). Let $\{u_n\}$ be a sequence is generated by Algorithms 2 and 3. For each $\omega^* \in VI(\Delta, \Gamma)$, we have*

$$\|z_n - \omega^*\|^2 \le \|u_n - \omega^*\|^2 - \left(1 - \chi^2\frac{\varkappa_n^2}{\varkappa_{n+1}^2}\right)\|u_n - v_n\|^2.$$

Proof. Let $\omega^* \in VI(\Delta,\Gamma)$ and by definition of z_n, we have

$$\begin{aligned}
\|z_n-\omega^*\|^2 &= \|v_n+\varkappa_n[\Gamma(u_n)-\Gamma(v_n)]-\omega^*\|^2\\
&= \|v_n-\omega^*\|^2+\varkappa_n^2\|\Gamma(u_n)-\Gamma(v_n)\|^2+2\varkappa_n\langle v_n-\omega^*,\Gamma(u_n)-\Gamma(v_n)\rangle\\
&= \|v_n+u_n-u_n-\omega^*\|^2+\varkappa_n^2\|\Gamma(u_n)-\Gamma(v_n)\|^2\\
&\quad +2\varkappa_n\langle v_n-\omega^*,\Gamma(u_n)-\Gamma(v_n)\rangle\\
&= \|v_n-u_n\|^2+\|u_n-\omega^*\|^2+2\langle v_n-u_n,u_n-\omega^*\rangle\\
&\quad +\varkappa_n^2\|\Gamma(u_n)-\Gamma(v_n)\|^2+2\varkappa_n\langle v_n-\omega^*,\Gamma(u_n)-\Gamma(v_n)\rangle\\
&= \|u_n-\omega^*\|^2+\|v_n-u_n\|^2+2\langle v_n-u_n,v_n-\omega^*\rangle+2\langle v_n-u_n,\\
&\quad u_n-v_n\rangle+\varkappa_n^2\|\Gamma(u_n)-\Gamma(v_n)\|^2+2\varkappa_n\langle v_n-\omega^*,\Gamma(u_n)-\Gamma(v_n)\rangle.
\end{aligned} \tag{3.13}$$

It is given that $v_n = P_\Delta[u_n-\varkappa_n\Gamma(u_n)]$ and indicates that

$$\langle u_n-\varkappa_n\Gamma(u_n)-v_n, v-v_n\rangle \le 0, \quad \forall v\in\Delta \tag{3.14}$$

or equivalently for some $\omega^*\in VI(\Delta,\Gamma)$, we can write

$$\langle u_n-v_n,\omega^*-v_n\rangle \le \varkappa_n\langle\Gamma(u_n),\omega^*-v_n\rangle. \tag{3.15}$$

Combining expressions (3.13) and (3.15), we have

$$\begin{aligned}
\|z_n-\omega^*\|^2 &\le \|u_n-\omega^*\|^2+\|v_n-u_n\|^2+2\varkappa_n\langle\Gamma(u_n),\omega^*-v_n\rangle-2\langle u_n-v_n,\\
&\quad u_n-v_n\rangle+\varkappa_n^2\|\Gamma(u_n)-\Gamma(v_n)\|^2-2\varkappa_n\langle\Gamma(u_n)-\Gamma(v_n),\omega^*-v_n\rangle\\
&= \|u_n-\omega^*\|^2-\|u_n-v_n\|^2+\varkappa_n^2\|\Gamma(u_n)-\Gamma(v_n)\|^2\\
&\quad -2\varkappa_n\langle\Gamma(v_n),v_n-\omega^*\rangle.
\end{aligned} \tag{3.16}$$

It is given that ω^* is the solution of the problem (VIP), which implies that

$$\langle\Gamma(\omega^*),v-\omega^*\rangle > 0, \quad \forall v\in\Delta.$$

Due to the property of Γ on Δ, we obtain

$$\langle\Gamma(v),v-\omega^*\rangle \ge 0, \quad \forall v\in\Delta.$$

Substituting $v=v_n\in\Delta$, we have

$$\langle\Gamma(v_n),v_n-\omega^*\rangle \ge 0. \tag{3.17}$$

Combining expressions (3.16) and (3.17), we obtain

$$\begin{aligned}
\|z_n-\omega^*\|^2 &\le \|u_n-\omega^*\|^2-\|u_n-v_n\|^2+\chi^2\frac{\varkappa_n^2}{\varkappa_{n+1}^2}\|u_n-v_n\|^2\\
&= \|u_n-\omega^*\|^2-\left(1-\chi^2\frac{\varkappa_n^2}{\varkappa_{n+1}^2}\right)\|u_n-v_n\|^2.
\end{aligned} \tag{3.18}$$

□

Theorem 3.1. *Assume that the conditions* (Γ1)–(Γ4) *are satisfied. Then, the sequence* $\{u_n\}$ *generated by Algorithm* 3 *converges strongly to an element* $\omega^* = P_{VI(\Delta,\Gamma)}(0)$.

Proof. Since $\varkappa_n \to \varkappa$ such that there exists a fixed number $\epsilon \in (0, 1-\chi^2)$ such that

$$\lim_{n\to+\infty}\left(1-\chi^2\frac{\varkappa_n^2}{\varkappa_{n+1}^2}\right) = 1-\chi^2 > \epsilon > 0.$$

Thus, expression (3.18) implies that

$$\|z_n - \omega^*\|^2 \le \|u_n - \omega^*\|^2, \quad \forall n \ge n_0. \tag{3.19}$$

It is given that $\omega^* \in VI(\Delta, \Gamma)$, we obtain

$$\begin{aligned}\|u_{n+1} - \omega^*\| &= \|(1-\alpha_n-\vartheta_n)u_n + \alpha_n z_n - \omega^*\| \\ &= \|(1-\alpha_n-\vartheta_n)(u_n-\omega^*) + \alpha_n(z_n-\omega^*) - \vartheta_n\omega^*\| \\ &\le \|(1-\alpha_n-\vartheta_n)(u_n-\omega^*) + \alpha_n(z_n-\omega^*)\| + \vartheta_n\|\omega^*\|. \end{aligned} \tag{3.20}$$

Following that, we calculate the following:

$$\begin{aligned}&\|(1-\alpha_n-\vartheta_n)(u_n-\omega^*) + \alpha_n(z_n-\omega^*)\|^2 \\ &\quad= (1-\alpha_n-\vartheta_n)^2\|u_n-\omega^*\|^2 + \alpha_n^2\|z_n-\omega^*\|^2 + 2\langle(1-\alpha_n-\vartheta_n) \\ &\qquad\times (u_n-\omega^*), \alpha_n(z_n-\omega^*)\rangle \\ &\quad\le (1-\alpha_n-\vartheta_n)^2\|u_n-\omega^*\|^2 + \alpha_n^2\|z_n-\omega^*\|^2 + 2\alpha_n(1-\alpha_n-\vartheta_n) \\ &\qquad\times \|u_n-\omega^*\|\|z_n-\omega^*\| \end{aligned} \tag{3.21}$$

$$\begin{aligned}&\quad\le (1-\alpha_n-\vartheta_n)^2\|u_n-\omega^*\|^2 + \alpha_n^2\|z_n-\omega^*\|^2 \\ &\qquad+ \alpha_n(1-\alpha_n-\vartheta_n)\|u_n-\omega^*\|^2 + \alpha_n(1-\alpha_n-\vartheta_n)\|z_n-\omega^*\|^2 \\ &\quad\le (1-\alpha_n-\vartheta_n)(1-\vartheta_n)\|u_n-\omega^*\|^2 + \alpha_n(1-\vartheta_n)\|z_n-\omega^*\|^2. \end{aligned} \tag{3.22}$$

Combining expressions (3.19) into (3.22), we get

$$\begin{aligned}&\|(1-\alpha_n-\vartheta_n)(u_n-\omega^*) + \alpha_n(z_n-\omega^*)\|^2 \\ &\quad\le (1-\alpha_n-\vartheta_n)(1-\vartheta_n)\|u_n-\omega^*\|^2 + \alpha_n(1-\vartheta_n)\|u_n-\omega^*\|^2 \\ &\quad= (1-\vartheta_n)^2\|u_n-\omega^*\|^2. \end{aligned} \tag{3.23}$$

Therefore, we have to calculate the following:

$$\|(1-\alpha_n-\vartheta_n)(u_n-\omega^*) + \alpha_n(z_n-\omega^*)\| \le (1-\vartheta_n)\|u_n-\omega^*\|. \tag{3.24}$$

Combining expressions (3.20) and (3.24), we have

$$\begin{aligned}\|u_{n+1}-\omega^*\| &\leq (1-\vartheta_n)\|u_n-\omega^*\|+\vartheta_n\|\omega^*\| \\ &\leq \max\{\|u_n-\omega^*\|, \|\omega^*\|\} \\ &\leq \max\{\|u_{n_0}-\omega^*\|, \|\omega^*\|\}. \end{aligned} \tag{3.25}$$

Thus, the above expression implies that $\{u_n\}$ is bounded sequence. Next, our aim is to prove that the sequence $\{u_n\}$ is strongly convergent. Indeed, by the use of definition of $\{u_{n+1}\}$, we have

$$\begin{aligned}\|u_{n+1}-\omega^*\|^2 &= \|(1-\alpha_n-\vartheta_n)u_n+\alpha_n z_n-\omega^*\|^2 \\ &= \|(1-\alpha_n-\vartheta_n)(u_n-\omega^*)+\alpha_n(z_n-\omega^*)-\vartheta_n\omega^*\|^2 \\ &= \|(1-\alpha_n-\vartheta_n)(u_n-\omega^*)+\alpha_n(z_n-\omega^*)\|^2+\vartheta_n^2\|\omega^*\|^2 \\ &\quad -2\langle(1-\alpha_n-\vartheta_n)(u_n-\omega^*)+\alpha_n(z_n-\omega^*), \vartheta_n\omega^*\rangle. \end{aligned} \tag{3.26}$$

By the use of expression (3.22), we have

$$\begin{aligned}&\|(1-\alpha_n-\vartheta_n)(u_n-\omega^*)+\alpha_n(z_n-\omega^*)\|^2 \\ &\quad \leq (1-\alpha_n-\vartheta_n)(1-\vartheta_n)\|u_n-\omega^*\|^2+\alpha_n(1-\vartheta_n)\|z_n-\omega^*\|^2. \end{aligned} \tag{3.27}$$

Combining expressions (3.26) and (3.27) (for some $K_2>0$), we have

$$\begin{aligned}\|u_{n+1}-\omega^*\|^2 &\leq (1-\alpha_n-\vartheta_n)(1-\vartheta_n)\|u_n-\omega^*\|^2+\alpha_n(1-\vartheta_n)\|z_n-\omega^*\|^2+\vartheta_n K_2 \\ &\leq (1-\alpha_n-\vartheta_n)(1-\vartheta_n)\|u_n-\omega^*\|^2+\vartheta_n K_2 \\ &\quad +\alpha_n(1-\vartheta_n)\left[\|u_n-\omega^*\|^2-\left(1-\chi^2\frac{\varkappa_n^2}{\varkappa_{n+1}^2}\right)\|u_n-v_n\|^2\right] \\ &= (1-\vartheta_n)^2\|u_n-\omega^*\|^2+\vartheta_n K_2 \\ &\quad -\alpha_n(1-\vartheta_n)\left[\left(1-\chi^2\frac{\varkappa_n^2}{\varkappa_{n+1}^2}\right)\|u_n-v_n\|^2\right] \\ &\leq \|u_n-\omega^*\|^2+\vartheta_n K_2-\alpha_n(1-\vartheta_n)\left[\left(1-\chi^2\frac{\varkappa_n^2}{\varkappa_{n+1}^2}\right)\|u_n-v_n\|^2\right]. \end{aligned} \tag{3.28}$$

By following the conditions (Γ1)–(Γ4), the solution set $VI(\Delta,\Gamma)$ is a closed and convex set. It is given that $\omega^*=P_{VI(\Delta,\Gamma)}(0)$, and by Lemma 2.2 (ii), we have

$$\langle 0-\omega^*, v-\omega^*\rangle \leq 0, \quad \forall v\in VI(\Delta,\Gamma). \tag{3.29}$$

Next, we divide the rest of the proof into the following two parts:

Case 1. Suppose that there exists a fixed number $n_1 \in \mathbb{N}$ such that

$$\|u_{n+1} - \omega^*\| \leq \|u_n - \omega^*\|, \quad \forall n \geq n_1. \tag{3.30}$$

Then $\lim_{n\to+\infty} \|u_n - \omega^*\|$ exists. From (3.28), we have

$$\alpha_n(1-\vartheta_n)\left[\left(1-\chi^2\frac{\varkappa_n^2}{\varkappa_{n+1}^2}\right)\|u_n - v_n\|^2\right] \leq \|u_n - \omega^*\|^2 + \vartheta_n K_2 - \|u_{n+1} - \omega^*\|^2. \tag{3.31}$$

The existence of $\lim_{n\to+\infty} \|u_n - \omega^*\|$ and $\vartheta_n \to 0$, we infer that

$$\lim_{n\to+\infty} \|u_n - v_n\| = 0. \tag{3.32}$$

It follows that

$$\|z_n - v_n\| = \|v_n + \varkappa_n[\Gamma(u_n) - \Gamma(v_n)] - v_n\| \leq \varkappa_0 L\|u_n - v_n\|.$$

The above expression implies that

$$\lim_{n\to+\infty} \|z_n - v_n\| = 0. \tag{3.33}$$

It follows that

$$\lim_{n\to+\infty} \|u_n - z_n\| \leq \lim_{n\to+\infty} \|u_n - v_n\| + \lim_{n\to+\infty} \|v_n - z_n\| = 0. \tag{3.34}$$

It follows from expression (3.34) and $\vartheta_n \to 0$, such that

$$\begin{aligned}\|u_{n+1} - u_n\| &= \|(1-\alpha_n-\vartheta_n)u_n + \alpha_n z_n - u_n\| \\ &= \|u_n - \vartheta_n u_n + \alpha_n z_n - \alpha_n u_n - u_n\| \\ &\leq \alpha_n\|z_n - u_n\| + \vartheta_n\|u_n\|,\end{aligned} \tag{3.35}$$

which gives that

$$\|u_{n+1} - u_n\| \to 0 \quad \text{as } n \to +\infty. \tag{3.36}$$

We can also deduce that $\{v_n\}$ and $\{z_n\}$ are bounded. The reflexivity of Σ and the boundedness of $\{u_n\}$ guarantee that there exists a subsequence $\{u_{n_k}\}$ such that $\{u_{n_k}\} \rightharpoonup \hat{u} \in \Sigma$ as $k \to +\infty$. Since $\{u_{n_k}\}$ weakly convergent to $\hat{u}$ and due to $\lim_{k\to+\infty} \|u_{n_k} - v_{n_k}\| = 0$, the sequence $\{v_{n_k}\}$ also weakly convergent to $\hat{u}$. Next, we need to prove that $\hat{u} \in VI(\Delta, \Gamma)$. By value of v_n, we have

$$v_{n_k} = P_\Delta[u_{n_k} - \varkappa_{n_k}\Gamma(u_{n_k})]$$

that is equivalent to

$$\langle u_{n_k} - \varkappa_{n_k}\Gamma(u_{n_k}) - v_{n_k}, v - v_{n_k}\rangle \leq 0, \quad \forall v \in \Delta. \tag{3.37}$$

The above inequality implies that

$$\langle u_{n_k} - v_{n_k}, v - v_{n_k}\rangle \leq \varkappa_{n_k}\langle \Gamma(u_{n_k}), v - v_{n_k}\rangle, \quad \forall v \in \Delta. \tag{3.38}$$

Thus, we obtain

$$\begin{aligned}&\frac{1}{\varkappa_{n_k}}\langle u_{n_k} - v_{n_k}, v - v_{n_k}\rangle + \langle \Gamma(u_{n_k}), v_{n_k} - u_{n_k}\rangle\\ &\quad \leq \langle \Gamma(u_{n_k}), v - u_{n_k}\rangle, \quad \forall v \in \Delta.\end{aligned} \tag{3.39}$$

By the use of $\lim_{k\to+\infty} \|u_{n_k} - v_{n_k}\| = 0$ and $k \to +\infty$ in (3.39), we have

$$\liminf_{k\to+\infty}\langle \Gamma(u_{n_k}), v - u_{n_k}\rangle \geq 0, \quad \forall v \in \Delta. \tag{3.40}$$

Furthermore, it implies that

$$\begin{aligned}\langle \Gamma(v_{n_k}), v - v_{n_k}\rangle &= \langle \Gamma(v_{n_k}) - \Gamma(u_{n_k}), v - u_{n_k}\rangle + \langle \Gamma(u_{n_k}), v - u_{n_k}\rangle\\ &\quad + \langle \Gamma(v_{n_k}), u_{n_k} - v_{n_k}\rangle.\end{aligned} \tag{3.41}$$

Since $\lim_{k\to+\infty} \|u_{n_k} - v_{n_k}\| = 0$. Thus, we have

$$\lim_{k\to+\infty} \|\Gamma(u_{n_k}) - \Gamma(v_{n_k})\| = 0, \tag{3.42}$$

which together with (3.41) and (3.42), we obtain

$$\liminf_{k\to+\infty}\langle \Gamma(v_{n_k}), v - v_{n_k}\rangle \geq 0, \quad \forall v \in \Delta. \tag{3.43}$$

Moreover, let us take a positive sequence $\{\epsilon_k\}$ that is decreasing and convergent to zero. For each $\{\epsilon_k\}$ there exists a least positive integer denoted by m_k such that

$$\langle \Gamma(u_{n_i}), v - u_{n_i}\rangle + \epsilon_k > 0, \quad \forall i \geq m_k. \tag{3.44}$$

Since $\{\epsilon_k\}$ is decreasing sequence and it is easy to see that the sequence $\{m_k\}$ is increasing. If there exists a natural number $N_0 \in \mathbb{N}$ such that for all $\Gamma(u_{n_{m_k}}) \neq 0$, $n_{m_k} \geq N_0$. Consider that

$$\aleph_{n_{m_k}} = \frac{\Gamma(u_{n_{m_k}})}{\|\Gamma(u_{n_{m_k}})\|^2}, \quad \forall n_{m_k} \geq N_0. \tag{3.45}$$

Due to the above definition, we have

$$\langle \Gamma(u_{n_{m_k}}), \aleph_{n_{m_k}}\rangle = 1, \quad \forall n_{m_k} \geq N_0. \tag{3.46}$$

Moreover, from expressions (3.44) and (3.46) for all $n_{m_k} \geq N_0$, we have

$$\langle \Gamma(u_{n_{m_k}}), v + \epsilon_k \aleph_{n_{m_k}} - u_{n_{m_k}}\rangle > 0. \tag{3.47}$$

By the definition of quasimonotone, we have

$$\langle \Gamma(v + \epsilon_k \aleph_{n_{m_k}}), v + \epsilon_k \aleph_{n_{m_k}} - u_{n_{m_k}} \rangle > 0. \tag{3.48}$$

For all $n_{m_k} \geq N_0$, we have

$$\begin{aligned}\langle \Gamma(v), v - u_{n_{m_k}} \rangle \geq \langle \Gamma(v) - \Gamma(v + \epsilon_k \aleph_{n_{m_k}}), v + \epsilon_k \aleph_{n_{m_k}} - u_{n_{m_k}} \rangle \\ - \epsilon_k \langle \Gamma(v), \aleph_{n_{m_k}} \rangle.\end{aligned} \tag{3.49}$$

Due to $\{u_{n_k}\}$ converges weakly to $\hat{u} \in \Delta$ with Γ is weakly sequentially continuous on the set Δ we obtain $\{\Gamma(u_{n_k})\}$ converges weakly to $\Gamma(\hat{u})$. Let $\Gamma(\hat{u}) \neq 0$, which implies that

$$\|\Gamma(\hat{u})\| \leq \liminf_{k \to +\infty} \|\Gamma(u_{n_k})\|. \tag{3.50}$$

Since $\{u_{n_{m_k}}\} \subset \{u_{n_k}\}$ and $\lim_{k \to +\infty} \epsilon_k = 0$, we have

$$0 \leq \lim_{k \to +\infty} \|\epsilon_k \aleph_{n_{m_k}}\| = \lim_{k \to +\infty} \frac{\epsilon_k}{\|\Gamma(u_{n_{m_k}})\|} \leq \frac{0}{\|\Gamma(\hat{u})\|} = 0. \tag{3.51}$$

By letting $k \to +\infty$ in expression (3.49), we get

$$\langle \Gamma(v), v - \hat{u} \rangle \geq 0, \quad \forall v \in \Delta. \tag{3.52}$$

Let $u \in \Delta$ be arbitrary element and for $0 < \varkappa \leq 1$. Consider that

$$\hat{u}_{\varkappa} = \varkappa u + (1 - \varkappa)\hat{u}. \tag{3.53}$$

Then $\hat{u}_{\varkappa} \in \Delta$ and from expression (3.52), we have

$$\varkappa \langle \Gamma(\hat{u}_{\varkappa}), u - \hat{u} \rangle \geq 0. \tag{3.54}$$

Hence, we have

$$\langle \Gamma(\hat{u}_{\varkappa}), u - \hat{u} \rangle \geq 0. \tag{3.55}$$

Let $\varkappa \to 0$. Then $\hat{u}_{\varkappa} \to \hat{u}$ along a line segment. By the continuity of an operator, $\Gamma(\hat{u}_{\varkappa})$ converges to $\Gamma(\hat{u})$ as $\varkappa \to 0$. It follows from expression (3.55) such that

$$\langle \Gamma(\hat{u}), u - \hat{u} \rangle \geq 0. \tag{3.56}$$

Therefore $\hat{u}$ is a solution of problem (VIP). Furthermore, we have

$$\limsup_{n \to +\infty} \langle \omega^*, \omega^* - u_n \rangle = \limsup_{k \to +\infty} \langle \omega^*, \omega^* - u_{n_k} \rangle = \langle \omega^*, \omega^* - \hat{u} \rangle \leq 0. \tag{3.57}$$

By the use of $\lim_{n \to +\infty} \|u_{n+1} - u_n\| = 0$. We might conclude that

$$\limsup_{n \to +\infty} \langle \omega^*, \omega^* - u_{n+1} \rangle \leq \limsup_{n \to +\infty} \langle \omega^*, \omega^* - u_n \rangle + \limsup_{n \to +\infty} \langle \omega^*, u_n - u_{n+1} \rangle \leq 0. \tag{3.58}$$

Next, assume that $t_n = (1-\alpha_n)u_n + \alpha_n z_n$. Thus, we obtain

$$\begin{aligned} u_{n+1} &= t_n - \vartheta_n u_n = (1-\vartheta_n)t_n - \vartheta_n(u_n - t_n) \\ &= (1-\vartheta_n)t_n - \vartheta_n\alpha_n(u_n - z_n), \end{aligned} \tag{3.59}$$

where $u_n - t_n = u_n - (1-\alpha_n)u_n - \alpha_n z_n = \alpha_n(u_n - z_n)$. Thus, we have

$$\begin{aligned} \|u_{n+1} - \omega^*\|^2 &= \|(1-\vartheta_n)t_n + \alpha_n\vartheta_n(z_n - u_n) - \omega^*\|^2 \\ &= \|(1-\vartheta_n)(t_n - \omega^*) + [\alpha_n\vartheta_n(z_n - u_n) - \vartheta_n\omega^*]\|^2 \\ &\le (1-\vartheta_n)^2\|t_n - \omega^*\|^2 + 2\langle \alpha_n\vartheta_n(z_n - u_n) - \vartheta_n\omega^*, \\ &\quad (1-\vartheta_n)(t_n - \omega^*) + \alpha_n\vartheta_n(z_n - u_n) - \vartheta_n\omega^*\rangle \\ &= (1-\vartheta_n)^2\|t_n - \omega^*\|^2 + 2\langle \alpha_n\vartheta_n(z_n - u_n) - \vartheta_n\omega^*, t_n - \vartheta_n t_n \\ &\quad - \vartheta_n(u_n - t_n) - \omega^*\rangle \\ &= (1-\vartheta_n)\|t_n - \omega^*\|^2 + 2\alpha_n\vartheta_n\langle z_n - u_n, u_{n+1} - \omega^*\rangle \\ &\quad + 2\vartheta_n\langle \omega^*, \omega^* - u_{n+1}\rangle \\ &\le (1-\vartheta_n)\|t_n - \omega^*\|^2 + 2\alpha_n\vartheta_n\|z_n - u_n\|\|u_{n+1} - \omega^*\| \\ &\quad + 2\vartheta_n\langle \omega^*, \omega^* - u_{n+1}\rangle. \end{aligned} \tag{3.60}$$

Next, we need to evaluate

$$\begin{aligned} \|t_n - \omega^*\|^2 &= \|(1-\alpha_n)u_n + \alpha_n z_n - \omega^*\|^2 \\ &= \|(1-\alpha_n)(u_n - \omega^*) + \alpha_n(z_n - \omega^*)\|^2 \\ &= (1-\alpha_n)^2\|u_n - \omega^*\|^2 + \alpha_n^2\|z_n - \omega^*\|^2 + 2\langle (1-\alpha_n)(u_n - \omega^*), \\ &\quad \alpha_n(z_n - \omega^*)\rangle \\ &\le (1-\alpha_n)^2\|u_n - \omega^*\|^2 + \alpha_n^2\|z_n - \omega^*\|^2 + 2\alpha_n(1-\alpha_n)\|u_n - \omega^*\| \\ &\quad \times \|z_n - \omega^*\| \\ &\le (1-\alpha_n)^2\|u_n - \omega^*\|^2 + \alpha_n^2\|z_n - \omega^*\|^2 + \alpha_n(1-\alpha_n)\|u_n - \omega^*\|^2 \\ &\quad + \alpha_n(1-\alpha_n)\|z_n - \omega^*\|^2 \\ &= (1-\alpha_n)\|u_n - \omega^*\|^2 + \alpha_n\|z_n - \omega^*\|^2 \\ &\le (1-\alpha_n)\|u_n - \omega^*\|^2 + \alpha_n\|u_n - \omega^*\|^2 \\ &= \|u_n - \omega^*\|^2. \end{aligned} \tag{3.61}$$

Combining expressions (3.60) and (3.61) gives that

$$\begin{aligned} \|u_{n+1} - \omega^*\|^2 \le (1-\vartheta_n)\|u_n - \omega^*\|^2 + \vartheta_n[2\alpha_n\|z_n - u_n\|\|u_{n+1} - \omega^*\| \\ + 2\vartheta_n\langle \omega^*, \omega^* - u_{n+1}\rangle]. \end{aligned} \tag{3.62}$$

By the use of expressions (3.58), (3.62) and Lemma 2.3, we can derive that $\|u_n - \omega^*\| \to 0$ as $n \to +\infty$.

Case 2. Assume that there is a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\|u_{n_i} - \omega^*\| \leq \|u_{n_{i+1}} - \omega^*\|, \quad \forall i \in \mathbb{N}.$$

By the use of Lemma 2.4, there exists a sequence $\{m_k\} \subset \mathbb{N}$ ($\{m_k\} \to +\infty$), such that

$$\|u_{m_k} - \omega^*\| \leq \|u_{m_{k+1}} - \omega^*\| \quad \text{and} \quad \|u_k - \omega^*\| \leq \|u_{m_{k+1}} - \omega^*\|, \quad \forall k \in \mathbb{N}. \tag{3.63}$$

By the use of expression (3.31), we have

$$\begin{aligned} \alpha_{m_k}(1 - \vartheta_{m_k}) &\left[\left(1 - \frac{\chi^2 \varkappa_{m_k}^2}{\varkappa_{m_k+1}^2}\right) \|u_{m_k} - v_{m_k}\|^2\right] \\ &\leq \|u_{m_k} - \omega^*\|^2 + \vartheta_{m_k} K_2 - \|u_{m_k+1} - \omega^*\|^2. \end{aligned} \tag{3.64}$$

Due to $\vartheta_{m_k} \to 0$, we can deduce the following:

$$\lim_{n \to +\infty} \|u_{m_k} - v_{m_k}\| = 0. \tag{3.65}$$

It continues from that

$$\begin{aligned} \|u_{m_k+1} - u_{m_k}\| &= \|(1 - \alpha_{m_k} - \vartheta_{m_k})u_{m_k} + \alpha_{m_k} z_{m_k} - u_{m_k}\| \\ &= \|u_{m_k} - \vartheta_{m_k} u_{m_k} + \alpha_{m_k} z_{m_k} - \alpha_{m_k} u_{m_k} - u_{m_k}\| \\ &\leq \alpha_{m_k}\|z_{m_k} - u_{m_k}\| + \vartheta_{m_k}\|u_{m_k}\| \longrightarrow 0. \end{aligned} \tag{3.66}$$

By using similar argument as in Case 1, we get

$$\limsup_{k \to +\infty} \langle \omega^*, u_{m_k+1} - \omega^* \rangle \leq 0. \tag{3.67}$$

By the use of expressions (3.62) and (3.63), we have

$$\begin{aligned} \|u_{m_k+1} - \omega^*\|^2 &\leq (1 - \vartheta_{m_k})\|u_{m_k} - \omega^*\|^2 + \vartheta_{m_k}[2\alpha_{m_k}\|z_{m_k} - u_{m_k}\| \\ &\quad \times \|u_{m_k+1} - \omega^*\| + 2\vartheta_{m_k}\langle \omega^*, \omega^* - u_{m_k+1}\rangle] \\ &\leq (1 - \vartheta_{m_k})\|u_{m_{k+1}} - \omega^*\|^2 + \vartheta_{m_k}\Big[2\alpha_{m_k}\|z_{m_k} - u_{m_k}\| \\ &\quad \times \|u_{m_k+1} - \omega^*\| + 2\vartheta_{m_k}\langle \omega^*, \omega^* - u_{m_k+1}\rangle\Big]. \end{aligned} \tag{3.68}$$

It follows that

$$\|u_{m_k+1} - \omega^*\|^2 \leq 2\alpha_{m_k}\|z_{m_k} - u_{m_k}\|\|u_{m_k+1} - \omega^*\| + 2\vartheta_{m_k}\langle \omega^*, \omega^* - u_{m_k+1}\rangle. \tag{3.69}$$

Since $\vartheta_{m_k} \to 0$ and $\|u_{m_k} - \omega^*\|$ is bounded, (3.67) and (3.69), yield

$$\|u_{m_k+1} - \omega^*\|^2 \to 0, \quad \text{as } k \to +\infty. \tag{3.70}$$

The above implies that

$$\lim_{n\to+\infty} \|u_k - \omega^*\|^2 \le \lim_{n\to+\infty} \|u_{m_k+1} - \omega^*\|^2 \le 0. \tag{3.71}$$

As a result, $u_n \to \omega^*$ and the desired result will be obtained. This completes the proof of the theorem. □

4. Numerical Experiment

This section describes the numerical performance of the proposed algorithms, in contrast to some related work in the literature, as well as the analysis of how variations in control parameters affect the numerical effectiveness of the proposed algorithms. All computations are done in MATLAB R2018b and run on HP i-5 Core(TM)i5-6200 8.00 GB (7.78 GB usable) RAM laptop.

Example 4.1. Let $\Sigma = l_2$ be a real Hilbert space with with the sequences of real numbers satisfying the following condition:

$$|u_1|^2 + |u_2|^2 + \cdots + |u_n|^2 + \cdots < +\infty. \tag{4.1}$$

Assume that a mapping $\Gamma : \Delta \to \Delta$ is defined by

$$\Gamma(u) = (5 - \|u\|)u, \quad \forall u \in \Sigma,$$

where $\Delta = \{u \in \Sigma : \|u\| \le 3\}$. We can easily seen that Γ is weakly sequentially continuous on Σ and the solution set is $VI(\Delta, \Gamma) = \{0\}$. For any $u, v \in \Sigma$, we have

$$\begin{aligned}
\|\Gamma(u) - \Gamma(v)\| &= \|(5 - \|u\|)u - (5 - \|v\|)v\| \\
&= \|5(u - v) - \|u\|(u - v) - (\|u\| - \|v\|)v\| \\
&\le 5\|u - v\| + \|u\|\|u - v\| + |\|u\| - \|v\||\|v\| \\
&\le 5\|u - v\| + 3\|u - v\| + 3\|u - v\| \\
&\le 11\|u - v\|.
\end{aligned} \tag{4.2}$$

Hence Γ is L-Lipschitz continuous with $L = 11$. For any $u, v \in \Sigma$ and let $\langle \Gamma(u), v - u\rangle > 0$, such that

$$(5 - \|u\|)\langle u, v - u\rangle > 0.$$

Since $\|u\| \le 3$ and it implies that

$$\langle u, v - u\rangle > 0.$$

Consider that

$$\begin{aligned}
\langle \Gamma(v), v - u\rangle &= (5 - \|v\|)\langle v, v - u\rangle \\
&\ge (5 - \|v\|)\langle v, v - u\rangle - (5 - \|v\|)\langle u, v - u\rangle \\
&\ge 2\|u - v\|^2 \ge 0.
\end{aligned} \tag{4.3}$$

Table 1. Numerical results for Example 4.1.

u_1	Number of iterations			Execution time in seconds		
	Algorithm 1	Algorithm 2	Algorithm 3	Algorithm 1	Algorithm 2	Algorithm 3
$(3, 3, \ldots, 3_{50000}, 0, 0, \ldots)$	55	53	47	5.5476213	4.9985636	4.6573635
$(1, 2, \ldots, 50000, 0, 0, \ldots)$	61	54	43	7.3546281	6.9900636	5.9001837
$(12, 12, \ldots, 12_{50000}, 0, 0, \ldots)$	67	56	49	8.9975637	7.6583600	7.0013355
$(19, 19, \ldots, 19_{50000}, 0, 0, \ldots)$	73	55	51	10.658300	9.6501931	8.9857385
$(30, 30, \ldots, 30_{50000}, 0, 0, \ldots)$	89	67	61	12.758372	11.0018247	10.0957385

Hence a mapping Γ is quasimonotone on Δ. Let $u = (\frac{5}{2}, 0, 0, \cdots, 0, \cdots)$ and $v = (3, 0, 0, \cdots, 0, \cdots)$ such that

$$\langle \Gamma(u) - \Gamma(v), u - v \rangle = \left(\frac{5}{2} - 3\right)^3 < 0.$$

Let us consider the following projection formula:

$$P_\Delta(u) = \begin{cases} u & \text{if } \|u\| \leq 3, \\ \dfrac{3u}{\|u\|}, & \text{otherwise.} \end{cases}$$

Table 1 shows numerical results. The control conditions are taken in the following way:

(i) Algorithm 1:

$$\varkappa = \frac{0.7}{L}, \quad \vartheta_n = \frac{1}{(n+2)}, \quad \alpha_n = \frac{1}{2}(1 - \vartheta_n), \quad D_n = \|u_{n+1} - u_n\|;$$

(ii) Algorithm 2:

$$\varkappa_1 = 0.22, \quad \chi = 0.44, \quad \vartheta_n = \frac{1}{(n+2)},$$

$$\alpha_n = \frac{1}{2}(1 - \vartheta_n), \quad D_n = \|u_{n+1} - u_n\|;$$

(iii) Algorithm 3:

$$\varkappa_1 = 0.22, \quad \chi = 0.44, \quad \varphi_n = \frac{100}{(n+1)^2}, \quad \vartheta_n = \frac{1}{(n+2)},$$

$$\alpha_n = \frac{1}{2}(1 - \vartheta_n), \quad D_n = \|u_{n+1} - u_n\|.$$

5. Future Work

It could be interesting to look at the same algorithms and study their convergence in the context of uniformly convex Banach space.

6. Conclusion

To provide a numerical solution to quasi-monotone variational inequality problems in real Hilbert space, we developed several modified extragradient-type methods. Despite the fact that each sequence is produced by a different step size rule, all sequences produced by the proposed method are strongly convergent to the solution. The computational results are developed to illustrate the mathematical efficiency of our methods. These numerical studies have shown that the variable step size has an effect on the efficiency of the iterative sequence in this respect.

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