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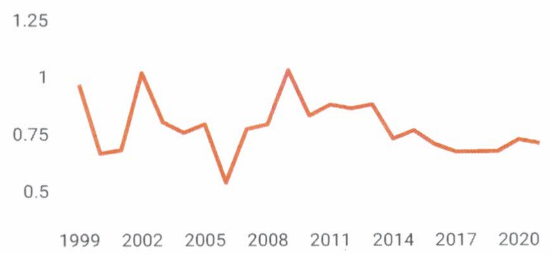

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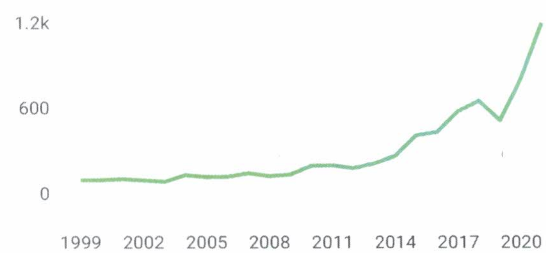

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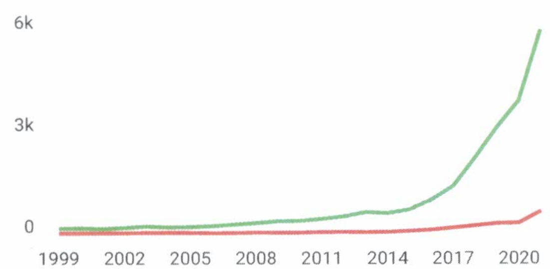

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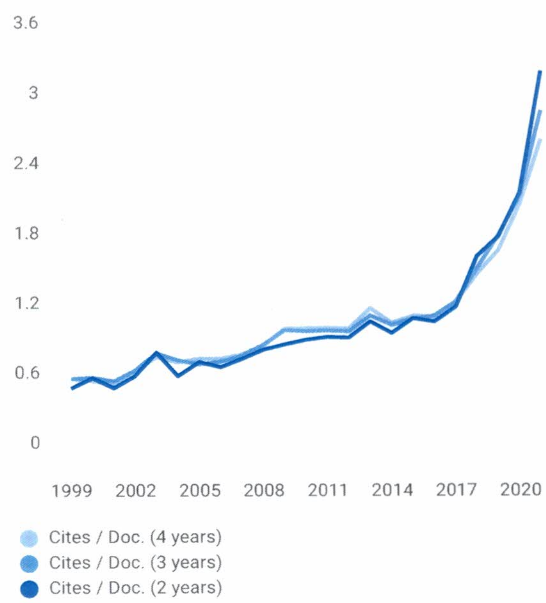

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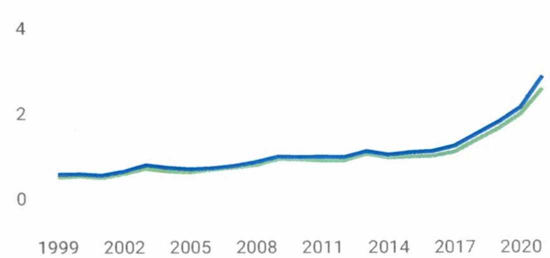

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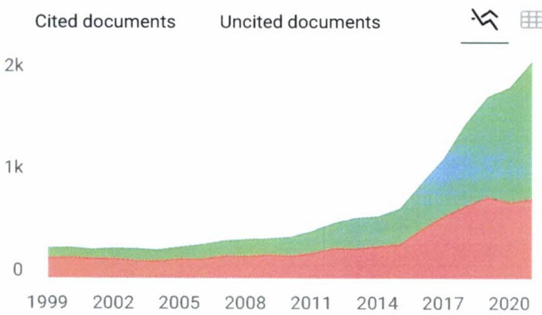

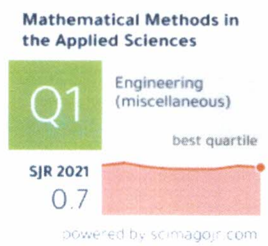

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RESEARCH ARTICLE

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Another hybrid approach for solving monotone operator equations and application to signal processing

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This paper presents a hybrid conjugate gradient (CG) approach for solving nonlinear equations and signal reconstruction. The CG parameter of the approach is a convex combination of the Dai-Yuan (DY)-like and Hestenes-Stiefel (HS)-like parameters. Independent of any line search, the search direction is descent and bounded. Under some reasonable assumptions, the global convergence of the hybrid approach is proved. Numerical experiments on some benchmark test problems show that the proposed approach is efficient compared with some existing algorithms. Finally, the proposed approach is applied in signal reconstruction.

KEYWORDS

compressive sensing, global convergence, gradient method, projection method, signal reconstruction, ℓ_1-norm minimization

MSC CLASSIFICATION

90C30; 65J15; 65K05; 65K10

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1 | INTRODUCTION

Let B be a nonempty closed and convex subset of $\mathbb{R}^n$ and $G : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous and monotone mapping. In this paper, the problem of interest is solving systems of nonlinear equations of the following form:

$$G(t) = 0,\ t \in B. \tag{1}$$

The appearance of problem (1) in various field of research and applications such as power flow equation, chemical equilibrium systems, compressive sensing,[1–5] and several others[6–9] have motivated several studies on effective methods for finding solutions to (1).

By choosing a good starting point, several iterative methods such as the Newton method, quasi-Newton method, Gauss-Newton method, Levenberg Marquardt method, and the trust region method[10–14] have shown to be efficient for solving (1). These methods exhibit fast convergence which makes them ideal choices for solving (1). A draw back associated with these methods which makes them not suitable for solving (1) when the dimension is large is the fact that they need to solve linear equations using the Jacobian matrix or an approximation of it per iteration.

The conjugate gradient method being one of the first-order optimization methods well known for solving large-scale unconstrained optimization problems due to its simplicity and low storage has been extended to solve (1) using the projection method of Solodov and Svaiter.[15] As a consequence, several other researchers have extended conjugate gradient methods for solving unconstrained optimization problems to solve (1). For instance, Cheng[16] extended the classical PRP method to solve system of monotone equations by combining it with the projection method. Ibrahim et al[17,18] extended the hybrid conjugate gradient method of Djordjević[19] to solve (1). Abubakar et al[20] also proposed a derivative-free iterative method for solving (1). The method uses a convex combination of the Hestenes-Stiefel (HS)[21] and the Dai-Yuan (DY) methods.[22] The method can be viewed as an extension of the hybrid conjugate gradient method for unconstrained optimization problem proposed in Andrei.[23] Beside the aforementioned articles, several other articles have focused on incorporating the conjugate gradient method for unconstrained optimization problem with the projection method to develop derivative-free methods for solving (1). Interested readers can refer to the following articles and references therein.[24–35]

Our interest in this article is to propose, analyze, and test a derivative-free hybrid iterative method for solving (1). The propose method also uses a convex combination of the HS and the DY methods as in Abubakar et al.[20] However, a different hybridization parameter is chosen in this paper. The method also incorporates the projection method and can be viewed as another extension of the HS and DY hybrid method proposed in Andrei.[23] Furthermore, the proposed method is suitable to solve large-scale non-smooth equations because of its low storage and only require the image of the function G under consideration. The global convergence is proved under the assumptions that the mapping G is monotone and Lipschitz continuous.

The remaining part of this paper is organized as follows. In Section 2, the proposed algorithm is described. In Section 3, the global convergence is established, and numerical results are reported in Sections 4 and 5. The paper is concluded in Section 6.

2 | ALGORITHM

In this section, we propose a derivative-free conjugate gradient projection based algorithm to solve (1). For a good understanding of the motivation, we recall the algorithm proposed by Andrei[36] for finding solution to the unconstrained optimization problem:

$$\min\{f(t) : t \in \mathbb{R}^n\}, \tag{2}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable real valued function with gradient at $t^{(k)}$ denoted by $g^{(k)} := \nabla f^{(k)}$. The algorithm proposed in Andrei[36] produces a sequence $\{t^{(k)}\}$ via the following iterative formula:

$$t^{(k+1)} = t^{(k)} + \alpha^{(k)} d^{(k)}, \ k = 0, 1, \cdots, \tag{3}$$

where $\{t^{(k)}\}$ is the previous point, $\{t^{(k+1)}\}$ is the current point, $\alpha^{(k)}$ is the step size, and $d^{(k)}$ is the search direction defined as

$$d^{(k)} := \begin{cases} -g^{(k)}, & \text{if } k = 0, \\ -g^{(k)} + \beta^{(k)} s^{(k-1)}, & \text{if } k \geq 1, \end{cases} \tag{4}$$

with $s^{(k)} = t^{k+1} - t^k$ and $\beta^{(k)}$ given as

$$\begin{aligned} \beta^{(k)} &:= \theta^{(k)} \beta_{HS}^{(k)} + (1 - \theta^{(k)}) \beta_{DY}^{(k)} \\ &:= \theta^{(k)} \frac{\langle g^{(k)}, y^{(k-1)} \rangle}{\langle d^{(k-1)}, y^{(k-1)} \rangle} + (1 - \theta^{(k)}) \frac{\|g^{(k)}\|^2}{\langle d^{(k-1)}, y^{(k-1)} \rangle}, \end{aligned} \tag{5}$$

$y^{(k-1)} = g^{(k)} - g^{(k-1)}$, and $\theta^{(k)}$ is a parameter satisfying $0 \leq \theta^{(k)} \leq 1$.

Inspired by the above parameter $\beta^{(k)}$ and letting $G^{(k)} := G(t^{(k)})$, we propose the following search direction:

$$d^{(k)} := \begin{cases} -G^{(k)}, & \text{if } k = 0, \\ -\left(1 + \beta^{(k)} \frac{\langle G^{(k)}, d^{(k-1)} \rangle}{\|G^{(k)}\|^2}\right) G^{(k)} + \beta^{(k)} d^{(k-1)}, & \text{if } k \geq 1, \end{cases} \tag{6}$$

where $\beta^{(k)}$ is given by

$$\begin{aligned} \beta^{(k)} &:= (1 - \theta^{(k)}) \beta_1^{(k)} + \theta^{(k)} \beta_2^{(k)} \\ &:= (1 - \theta^{(k)}) \frac{\langle G^{(k)}, y^{(k-1)} \rangle}{\langle d^{(k-1)}, u^{(k-1)} \rangle} + \theta^{(k)} \frac{\|G^{(k)}\|^2}{\langle d^{(k-1)}, u^{(k-1)} \rangle}, \end{aligned} \tag{7}$$

where

$$\theta^{(k)} := 1 - \frac{\langle G^{(k)}, d^{(k-1)} \rangle^2}{\|G^{(k)}\|^2 \|d^{(k-1)}\|^2} \in [0, 1], \ u^{(k-1)} := y^{(k-1)} + \eta^{(k-1)} d^{(k-1)}, \ y^{(k-1)} = G^{(k)} - G^{(k-1)},$$

$$\eta^{(k-1)} := 1 + \max\left\{0, -\frac{\langle d^{(k-1)}, y^{(k-1)} \rangle}{\|d^{(k-1)}\|^2}\right\}.$$

By the above modification, it is not difficult to see that

$$\langle d^{(k-1)}, u^{(k-1)} \rangle \geq \langle d^{(k-1)}, y^{(k-1)} \rangle + \|d^{(k-1)}\|^2 - \langle d^{(k-1)}, y^{(k-1)} \rangle = \|d^{(k-1)}\|^2 > 0. \tag{8}$$

Note that Equation (6) is defined such that the direction always satisfies the sufficient descent condition if $G^{(k)} := g^{(k)}$. In addition, the vector $y^{(k-1)}$ in (5) is replaced with $u^{(k-1)}$ in (7) such that $\beta_1^{(k)}$ and $\beta_2^{(k)}$ are well defined.

Remark 1. From the definition of $\beta^{(k)}$ and $\theta^{(k)}$ in (7), we have

$$|\beta^{(k)}| \leq |\beta_1^k| + |\beta_2^k|. \tag{9}$$

To describe the propose hybrid conjugate gradient algorithm, we first recall the projection map.

Definition 1. Let $B \subset \mathbb{R}^n$ be a nonempty closed convex set. Then, for any $x \in \mathbb{R}^n$, its projection onto B, denoted by $P_B(x)$, is defined by

$$P_B(x) := \arg\min\{\|x - y\| \; : \; y \in B\}.$$

A known property of P_B is that it is nonexpansive, that is,

$$\|P_B(x) - P_B(y)\| \leq \|x - y\|, \; \forall x, y \in \mathbb{R}^n. \tag{10}$$

In what follows, we present the steps of the derivative-free conjugate gradient algorithm. Throughout, we refer to the proposed algorithm as Algorithm 1.

Algorithm 1

Step 0. Select $t_0 \in \mathbb{R}^n$, parameters $\sigma > 0$, $\rho \in (0,1)$, $\gamma \in (0,2)$, $tol > 0$ and set $k := 0$.
Step 1. If $\|G^{(k)}\| \leq tol$, stop, otherwise go to **Step 2**.
Step 2. Compute $d^{(k)}$ by (6).
Step 3. Compute the step size $\alpha^{(k)} := \rho^i$ where i is the smallest non-negative integer such that

$$-\langle G(t^{(k)} + \alpha^{(k)} d^{(k)}), \; d^{(k)} \rangle \geq \sigma \alpha^{(k)} \|d^{(k)}\|^2. \tag{11}$$

Step 4. Compute

$$w^{(k+1)} := t^{(k)} + \alpha^{(k)} d^{(k)}. \tag{12}$$

If $w^{(k+1)} \in B$ and $\|G(w^{(k+1)})\| \leq tol$, stop. Else compute

$$t^{(k+1)} := P_B[t^{(k)} - \gamma \zeta_k G(w^{(k+1)})], \tag{13}$$

where

$$\zeta_k := \frac{\langle G(w^{(k+1)}), \; t^{(k)} - w^{(k+1)} \rangle}{\|G(w^{(k+1)})\|^2}.$$

Step 5. Let $k := k + 1$ and go to **Step 1**.

Remark 2. The parameter γ in (13) is chosen in the interval $(0,2)$ in order to show that the sequence $\{\|t^{(k)} - \bar{t}\|\}$ is non-increasing (see Lemma 4). In addition, the parameter γ plays a significant role in the numerical efficiency of Algorithm 1.

3 | CONVERGENCE ANALYSIS

To establish the convergence of Algorithm 1, We begin with the following assumptions:

N_1 The function G is monotone, that is,

$$\langle G(x) - G(y), (x - y) \rangle \geq 0, \forall x, y \in R^n.$$

N_2 The function G is Lipschitz continuous; that is, there exists a positive constant L such that

$$\|G(x) - G(y)\| \leq L\|x - y\|, \forall x, y \in R^n.$$

N_3 The solution set of problem (1) denoted by B' is nonempty.
N_4 $G^{(k)} \neq 0$ unless the solution of (1) is obtained.

Lemma 1. *Let $d^{(k)}$ be defined by (6); then, $d^{(k)}$ satisfies the sufficient descent condition. That is,*

$$\langle G^{(k)}, d^{(k)} \rangle = -\|G^{(k)}\|^2. \tag{14}$$

Proof. For $k = 0$, $\langle G_0, d_0 \rangle = -\|G_0\|^2$.
For $k \geq 1$, by (6),

$$\begin{aligned} \langle G^{(k)}, d^{(k)} \rangle &= -\left(1 + \beta^{(k)} \frac{\langle G^{(k)}, d^{(k-1)} \rangle}{\|G^{(k)}\|^2}\right) \langle G^{(k)}, G^{(k)} \rangle + \beta^{(k)} \langle G^{(k)}, d^{(k-1)} \rangle \\ &= -\|G^{(k)}\|^2 - \beta^{(k)} \frac{\langle G^{(k)}, d^{(k-1)} \rangle}{\|G^{(k)}\|^2} \|G^{(k)}\|^2 + \beta^{(k)} \langle G^{(k)}, d^{(k-1)} \rangle \\ &= -\|G^{(k)}\|^2 - \beta^{(k)} \langle G^{(k)}, d^{(k-1)} \rangle + \beta^{(k)} \langle G^{(k)}, d^{(k-1)} \rangle \\ &= -\|G^{(k)}\|^2. \end{aligned} \tag{15}$$

Remark 3. From (14), applying Cauchy-Schwartz inequality, we have

$$\|d^{(k)}\| \geq \|G^{(k)}\|. \tag{16}$$

□

The following lemma shows that Algorithm 1 is well defined.

Lemma 2. *Suppose N_2 hold. Then, there exists a step-length $\alpha^{(k)} = \rho^i$ satisfying (11) for some $i \in \mathbb{N} \cup \{0\}$ and $\forall k \geq 0$.*

Proof. Suppose there exists $k_0 \geq 0$ such that (11) does not hold for any non-negative integer i, i.e.,

$$-\langle G(t^{(k_0)} + \rho^i d^{(k_0)}), d^{(k_0)} \rangle < \sigma \rho^i \|d^{(k_0)}\|^2.$$

By N_2 and allowing $i \to \infty$, we get

$$-\langle G(t^{(k_0)}), d^{(k_0)} \rangle \leq 0. \tag{17}$$

Also from (15), we have

$$-\langle G(t^{(k_0)}), d^{(k_0)} \rangle \geq \|G(t^{(k_0)})\|^2 > 0,$$

which contradicts (17). The proof is complete. □

Lemma 3. *Suppose N_2 hold. If $\{w^{(k+1)}\}$ and $\{t^{(k)}\}$ are defined by (12) and (13) in Algorithm 1; then,*

$$\alpha^{(k)} > \min\left\{1, \frac{\rho\|G^{(k)}\|^2}{(L+\sigma)\|d^{(k)}\|^2}\right\}. \tag{18}$$

Proof. From the line search (11), if $\alpha^{(k)} \neq 1$, then $(\alpha^{(k)})' = \alpha^{(k)}\rho^{-1}$ does not satisfy (11), that is,

$$-\langle G(t^{(k)} + (\alpha^{(k)})' d^{(k)}), d^{(k)} \rangle < \sigma (\alpha^{(k)})' \|d^{(k)}\|^2.$$

Using (14) and N_2, we have

$$\begin{aligned} \|G^{(k)}\|^2 &\leq -\langle G^{(k)}, d^{(k)} \rangle \\ &= \langle G(t^{(k)} + (\alpha^{(k)})' d^{(k)}) - G^{(k)}, d^{(k)} \rangle - \langle G(t^{(k)} + (\alpha^{(k)})' d^{(k)}), d^{(k)} \rangle \\ &< (\alpha^{(k)})'(L+\sigma)\|d^{(k)}\|^2. \end{aligned}$$

Solving the above inequality for $(\alpha^{(k)})'$, the desired result is obtained. □

Lemma 4. *Let N_1-N_3 be fulfilled. If $\{w^{(k+1)}\}$ and $\{t^{(k)}\}$ are sequences defined by (12) and (13) in Algorithm 1, then $\{w^{(k+1)}\}$ and $\{t^{(k)}\}$ are bounded. Furthermore,*

$$\lim_{k\to\infty} \|t^{(k)} - w^{(k+1)}\| = 0. \tag{19}$$

Proof. We begin by showing that the sequences $\{t^{(k)}\}$ and $\{w^{(k+1)}\}$ are bounded. Suppose $\bar{t} \in B'$; then, by monotonicity of G, we get

$$\langle G(w^{(k+1)}), t^{(k)} - \bar{t}\rangle \geq \langle G(w^{(k+1)}), t^{(k)} - w^{(k+1)}\rangle. \tag{20}$$

From the definition of $w^{(k+1)}$ and (11), we have

$$\langle G(w^{(k+1)}), t^{(k)} - w^{(k+1)}\rangle \geq \sigma(\alpha^{(k)})^2\|d^{(k)}\|^2 \geq 0. \tag{21}$$

Consequently, by (10), (20), (21), the definition of ζ_k, and $\gamma \in (0,2)$, we have

$$\begin{aligned}
\|t^{(k+1)} - \bar{t}\|^2 &= \|P_B[t^{(k)} - \gamma\zeta_k G(w^{(k+1)})] - P_B(\bar{t})\|^2 \\
&\leq \|t^{(k)} - \gamma\zeta_k G(w^{(k+1)}) - \bar{t}\|^2 \\
&= \|t^{(k)} - \bar{t}\|^2 - 2\gamma\zeta_k\langle G(w^{(k+1)}), t^{(k)} - \bar{t}\rangle + \|\gamma\zeta_k G(w^{(k+1)})\|^2 \\
&= \|t^{(k)} - \bar{t}\|^2 - 2\gamma\frac{\langle G(w^{(k+1)}), t^{(k)} - w^{(k+1)}\rangle}{\|G(w^{(k+1)})\|^2}\langle G(w^{(k+1)}), t^{(k)} - \bar{t}\rangle + \gamma^2\left(\frac{\langle G(w^{(k+1)}), t^{(k)} - w^{(k+1)}\rangle}{\|G(w^{(k+1)})\|}\right)^2 \\
&\leq \|t^{(k)} - \bar{t}\|^2 - 2\gamma\frac{\langle G(w^{(k+1)}), t^{(k)} - w^{(k+1)}\rangle}{\|G(w^{(k+1)})\|^2}\langle G(w^{(k+1)}), t^{(k)} - w^{(k+1)}\rangle + \gamma^2\left(\frac{\langle G(w^{(k+1)}), t^{(k)} - w^{(k+1)}\rangle}{\|G(w^{(k+1)})\|}\right)^2 \\
&\leq \|t^{(k)} - \bar{t}\|^2 - \gamma(2-\gamma)\left(\frac{\langle G(w^{(k+1)}), t^{(k)} - w^{(k+1)}\rangle}{\|G(w^{(k+1)})\|}\right)^2 \\
&\leq \|t^{(k)} - \bar{t}\|^2 - \gamma(2-\gamma)\frac{\sigma^2\|t^{(k)} - w^{(k+1)}\|^4}{\|G(w^{(k+1)})\|^2}.
\end{aligned} \tag{22}$$

Thus, the sequence $\{\|t^{(k)} - \bar{t}\|\}$ is non-increasing and convergent, and hence, $\{t^{(k)}\}$ is bounded. That is,

$$\|t^{(k)}\| \leq b,\ b > 0. \tag{23}$$

Moreover, from relation (22), we have

$$\|t^{(k+1)} - \bar{t}\|^2 \leq \|t^{(k)} - \bar{t}\|^2, \tag{24}$$

and we can deduce recursively that

$$\|t^{(k)} - \bar{t}\|^2 \leq \|t_0 - \bar{t}\|^2,\ \forall k \geq 0.$$

Thus, by N_2, we have that

$$\|G^{(k)}\| = \|G^{(k)} - G(\bar{t})\| \leq L\|t^{(k)} - \bar{t}\| \leq L\|t_0 - \bar{t}\|.$$

Letting $L\|t_0 - \bar{t}\| = A$, then the sequence $\{G^{(k)}\}$ is bounded. That is,

$$\|G^{(k)}\| \leq A,\ \forall k \geq 0. \tag{25}$$

By the definition of $w^{(k+1)}$, Cauchy-Schwartz inequality, N_1, and (21),

$$\sigma\|t^{(k)} - w^{(k+1)}\| = \frac{\sigma\|t^{(k)} - w^{(k+1)}\|^2}{\|t^{(k)} - w^{(k+1)}\|} = \frac{\sigma\|\alpha^{(k)}d^{(k)}\|^2}{\|t^{(k)} - w^{(k+1)}\|} \leq \frac{\langle G(w^{(k+1)}), t^{(k)} - w^{(k+1)}\rangle}{\|t^{(k)} - w^{(k+1)}\|} \leq \frac{\langle G^{(k)}, t^{(k)} - w^{(k+1)}\rangle}{\|t^{(k)} - w^{(k+1)}\|} \leq \|G^{(k)}\|. \tag{26}$$

By (26) and the reverse triangle inequality,

$$\sigma(\|w^{(k+1)}\| - \|t^{(k)}\|) \leq \sigma\|w^{(k+1)} - t^{(k)}\| \leq \|G^{(k)}\|.$$

The above relation together with (23) and (25) yields

$$\begin{aligned}\|w^{(k+1)}\| &\leq \frac{1}{\sigma}\|G^{(k)}\| + \|t^{(k)}\| \\ &\leq \frac{1}{\sigma}B + b.\end{aligned}$$

Therefore, the sequence $\{w^{(k+1)}\}$ is bounded.

Now, for any $\bar{t} \in B'$, the sequence $\{w^{(k+1)} - \bar{t}\}$ is also bounded; that is, there exists a positive constant $v > 0$ such that

$$\|w^{(k+1)} - \bar{t}\| \leq v, \forall k \geq 0.$$

The above inequality together with assumption N_2 yields

$$\|G(w^{(k+1)})\| = \|G(w^{(k+1)}) - G(\bar{t})\| \leq L\|w^{(k+1)} - \bar{t}\| \leq Lv.$$

Therefore, using relation (22), we have

$$\gamma(2-\gamma)\frac{\sigma^2}{(Lv)^2}\|t^{(k)} - w^{(k+1)}\|^4 \leq \|t^{(k)} - \bar{t}\|^2 - \|t^{(k+1)} - \bar{t}\|^2,$$

which implies

$$\gamma(2-\gamma)\frac{\sigma^2}{(Lv)^2}\sum_{k=0}^{\infty}\|t^{(k)} - w^{(k+1)}\|^4 \leq \sum_{k=0}^{\infty}(\|t^{(k)} - \bar{t}\|^2 - \|t^{(k+1)} - \bar{t}\|^2) \leq \|t^{(0)} - \bar{t}\| < \infty. \tag{27}$$

Relation (27) implies that

$$\lim_{k\to\infty}\|t^{(k)} - w^{(k+1)}\| = 0.$$

□

Remark 4. From (19) and the definition of $w^{(k+1)}$,

$$\lim_{k\to\infty}\alpha^{(k)}\|d^{(k)}\| = 0. \tag{28}$$

Theorem 1. *Let the sequence $\{t^{(k)}\}$ be generated by (13) in Algorithm 1; then,*

$$\liminf_{k\to\infty}\|G^{(k)}\| = 0. \tag{29}$$

Proof. Suppose by contradiction that (29) is not true, then there exists $r_0 > 0$ such that $\forall k \geq 0$,

$$\|G^{(k)}\| \geq r_0. \tag{30}$$

Inequality (16) together with (30) implies that

$$\|d^{(k)}\| \geq r_0, \forall k \geq 0. \tag{31}$$

Now, from (6), (9), (23), (25), (31), assumption N_2, and the Cauchy-Schwartz inequality,

$$\begin{aligned}
\|d^{(k)}\| &= \left\| -\left(1+\beta^{(k)}\frac{\langle G^{(k)}, d^{(k-1)}\rangle}{\|G^{(k)}\|^2}\right)G^{(k)} + \beta^{(k)}d^{(k-1)}\right\| \\
&\le \|G^{(k)}\| + |\beta^{(k)}|\frac{\|G^{(k)}\|^2\|d^{(k-1)}\|}{\|G^{(k)}\|^2} + |\beta^{(k)}|\|d^{(k-1)}\| \\
&\le \|G^{(k)}\| + 2\left(|\beta_1^{(k)}| + |\beta_2^{(2)}|\right)\|d^{(k-1)}\| \\
&= \|G^{(k)}\| + 2\left(\frac{\|G^{(k)}\|\|y^{(k-1)}\|}{\langle d^{(k-1)}, u^{(k-1)}\rangle} + \frac{\|G^{(k)}\|^2}{\langle d^{(k-1)}, u^{(k-1)}\rangle}\right)\|d^{(k-1)}\| \\
&\le \|G^{(k)}\| + 2\left(\frac{\|G^{(k)}\|\|y^{(k-1)}\|}{\|d^{(k-1)}\|^2} + \frac{\|G^{(k)}\|^2}{\|d^{(k-1)}\|^2}\right)\|d^{(k-1)}\| \\
&= \|G^{(k)}\| + 2\|G^{(k)}\|\left(\frac{\|y^{(k-1)}\| + \|G^{(k)}\|}{\|d^{(k-1)}\|}\right) \\
&= \left(1+\frac{2(\|y^{(k-1)}\| + \|G^{(k)}\|)}{\|d^{(k-1)}\|}\right)\|G^{(k)}\| \\
&\le \left(1+\frac{2(L\|t^{(k)} - t^{(k-1)}\| + \|G^{(k)}\|)}{\|d^{(k-1)}\|}\right)\|G^{(k)}\| \\
&\le \left(1+\frac{2(L(\|t^{(k)}\| + \|t^{(k-1)}\|) + \|G^{(k)}\|)}{\|d^{(k-1)}\|}\right)\|G^{(k)}\| \\
&\le \left(1+\frac{4bL+2A}{r_0}\right)A.
\end{aligned} \tag{32}$$

Setting $M = \left(1+\frac{4bL+2A}{r_0}\right)A$, we have

$$\|d^{(k)}\| \le M. \tag{33}$$

Multiplying both sides of (18) with $\|d^{(k)}\|$ together with (31) and (32),

$$\begin{aligned}
\alpha^{(k)}\|d^{(k)}\| &> \min\left\{1, \frac{\rho\|G^{(k)}\|^2}{(L+\sigma)\|d^{(k)}\|^2}\right\}\|d^{(k)}\| \\
&\ge \min\left\{r_0, \frac{\rho r_0^2}{(L+\sigma)M}\right\}.
\end{aligned}$$

The inequality above contradicts (28) and hence (29) holds. □

4 | NUMERICAL EXPERIMENTS ON SOME BENCHMARK TEST PROBLEMS

A traditional way to assess the efficiency of an algorithm is to compare its numerical strength with other algorithms based on some standard metrics. The standard metrics used in this experiment are number of iterations (ITER), number of function evaluations (FVAL), and CPU time (TIME). To assess the strength of Algorithm 1, we compare it with following algorithms:

- The algorithm called PRPFR proposed by Zhou et al.[37]
- The algorithm called HCGP proposed by Koorapetse and Kaelo.[38]

For the experiments, we use the following:

- Number of problems: 10.
- Five dimensions: 1,000, 5,000, 10,000, 50,000, 100,000.
- Five initial points: $x_1 = (0.1, 0.1, \cdots, 0.1)^T, x_2 = (0.2, 0.2, \cdots, 0.2)^T, x_3 = (0.5, 0.5, \cdots, 0.5)^T, x_4 = (1.5, 1.5, \cdots 1.5)^T$, $x_5 = (2, 2, \cdots, 2)^T$.

TABLE 1 Numerical results of Problem 1

DIM	INP	Algorithm 1 ITER	FVAL	TIME	NORM	PRPFR ITER	FVAL	TIME	NORM	HCGP ITER	FVAL	TIME	NORM
1,000	t_1	1	7	0.00445	0	22	67	0.004808	6.99E-06	14	43	0.012397	8.95E-06
	t_2	1	7	0.003296	0	23	70	0.005034	8.67E-06	15	46	0.006885	9.60E-06
	t_3	1	7	0.003247	0	25	76	0.004904	7.65E-06	16	49	0.004399	8.67E-06
	t_4	1	9	0.01108	0	78	236	0.01343	8.88E-06	1	5	0.002029	0
	t_5	1	10	0.002654	0	78	236	0.014869	9.12E-06	49	150	0.013952	9.44E-06
5,000	t_1	1	7	0.003758	0	22	67	0.014948	6.20E-06	13	40	0.008849	9.64E-06
	t_2	1	7	0.003712	0	23	70	0.014931	7.03E-06	14	43	0.011993	9.91E-06
	t_3	1	7	0.002996	0	24	73	0.01381	8.44E-06	15	46	0.00827	8.73E-06
	t_4	1	9	0.003333	0	76	230	0.039894	9.85E-06	1	5	0.002412	0
	t_5	1	10	0.003647	0	77	233	0.042804	8.85E-06	49	149	0.0274	8.82E-06
10,000	t_1	1	7	0.005903	0	22	67	0.024791	6.77E-06	13	40	0.014065	8.18E-06
	t_2	1	7	0.005237	0	23	70	0.027441	7.31E-06	14	43	0.016084	8.24E-06
	t_3	1	7	0.004847	0	24	73	0.028177	8.51E-06	14	44	0.015426	9.98E-06
	t_4	1	9	0.00598	0	76	230	0.076533	9.19E-06	1	5	0.00255	0
	t_5	1	10	0.006809	0	76	230	0.092785	9.43E-06	48	147	0.04935	9.48E-06
50,000	t_1	1	7	0.016805	0	23	70	0.081627	5.88E-06	12	38	0.048081	9.55E-06
	t_2	1	7	0.015145	0	24	73	0.084712	5.92E-06	13	41	0.048065	8.44E-06
	t_3	1	7	0.014309	0	25	76	0.090915	6.56E-06	14	43	0.051465	8.40E-06
	t_4	1	9	0.018584	0	75	227	0.25342	8.93E-06	1	5	0.007962	0
	t_5	1	10	0.021102	0	75	227	0.29542	9.17E-06	48	146	0.15137	8.87E-06
100,000	t_1	1	7	0.034895	0	23	70	0.19245	7.75E-06	12	38	0.087919	9.26E-06
	t_2	1	7	0.025437	0	24	73	0.17991	7.60E-06	13	41	0.095332	7.66E-06
	t_3	1	7	0.027235	0	25	76	0.18536	8.26E-06	14	43	0.10321	7.44E-06
	t_4	1	9	0.036193	0	74	224	0.58853	9.55E-06	1	5	0.014604	0
	t_5	1	10	0.046576	0	74	224	0.56748	9.80E-06	47	144	0.30941	9.54E-06

TABLE 2 Numerical results of Problem 2

		Algorithm 1				PRPFR				HCGP			
DIM	INP	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM
1,000	t_1	5	14	0.010907	5.13E-06	19	57	0.074332	6.77E-06	7	13	0.003686	2.24E-07
	t_2	5	14	0.002742	6.26E-06	20	60	0.004796	7.44E-06	7	13	0.002427	6.59E-06
	t_3	5	14	0.003355	5.62E-06	22	66	0.005223	6.04E-06	10	19	0.002903	4.16E-06
	t_4	5	14	0.003614	6.20E-06	23	68	0.006418	7.59E-06	8	15	0.002853	7.46E-07
	t_5	7	18	0.004825	3.75E-06	24	71	0.006396	7.53E-06	10	19	0.002738	2.93E-06
5,000	t_1	5	15	0.007452	2.18E-06	20	60	0.026726	7.46E-06	7	13	0.006304	4.81E-07
	t_2	5	15	0.006882	2.65E-06	21	63	0.018296	8.19E-06	8	16	0.006062	6.60E-06
	t_3	6	16	0.008752	2.31E-06	23	69	0.021802	6.65E-06	10	19	0.007671	9.72E-06
	t_4	6	16	0.00925	3.12E-06	24	71	0.021041	8.32E-06	8	15	0.00716	1.51E-06
	t_5	7	18	0.010331	7.82E-06	25	74	0.021078	8.24E-06	10	19	0.007987	6.24E-06
10,000	t_1	5	15	0.012364	3.05E-06	21	63	0.032699	5.27E-06	7	13	0.010527	6.76E-07
	t_2	5	15	0.013421	3.72E-06	22	66	0.033362	5.78E-06	8	16	0.011735	9.34E-06
	t_3	6	16	0.014644	3.23E-06	23	69	0.03473	9.38E-06	11	21	0.015886	3.04E-06
	t_4	6	16	0.014271	4.47E-06	25	74	0.044914	5.87E-06	8	15	0.010754	2.10E-06
	t_5	7	19	0.016597	2.20E-06	26	77	0.041468	5.81E-06	10	19	0.016068	8.77E-06
50,000	t_1	5	15	0.039806	6.77E-06	22	66	0.12089	5.88E-06	7	13	0.037796	1.50E-06
	t_2	5	15	0.040446	8.25E-06	23	69	0.16165	6.45E-06	9	17	0.047111	6.04E-07
	t_3	6	16	0.053597	7.16E-06	25	75	0.1369	5.23E-06	11	21	0.058209	6.81E-06
	t_4	6	17	0.048202	2.02E-06	26	77	0.13479	6.54E-06	8	15	0.043185	4.65E-06
	t_5	7	19	0.055566	4.87E-06	27	80	0.14756	6.48E-06	11	22	0.058472	8.18E-06
100,000	t_1	5	15	0.077451	9.56E-06	22	66	0.23511	8.31E-06	7	13	0.067591	2.13E-06
	t_2	6	17	0.083289	2.33E-06	23	69	0.24568	9.12E-06	9	17	0.086674	8.55E-07
	t_3	6	17	0.0895	2.02E-06	25	75	0.29601	7.40E-06	11	21	0.10511	9.62E-06
	t_4	6	17	0.097506	2.86E-06	26	77	0.28028	9.25E-06	8	15	0.078056	6.57E-06
	t_5	7	19	0.11362	6.88E-06	27	80	0.27247	9.17E-06	12	23	0.11552	6.73E-07

TABLE 3 Numerical results of Problem 3

		Algorithm 1				PRPFR				HCGP			
DIM	INP	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM
1,000	t_1	4	13	0.002685	7.77E-06	19	57	0.057757	6.02E-06	10	30	0.002653	6.00E-06
	t_2	5	15	0.002226	2.72E-06	20	60	0.006054	5.98E-06	10	31	0.002486	5.90E-06
	t_3	4	12	0.002295	2.51E-06	21	63	0.006013	7.14E-06	11	33	0.002545	6.57E-06
	t_4	11	35	0.005146	6.74E-06	22	66	0.008378	7.12E-06	12	36	0.00309	7.84E-06
	t_5	11	35	0.004623	6.74E-06	53	160	0.012471	8.98E-06	12	36	0.002888	7.84E-06
5,000	t_1	5	15	0.006869	3.48E-06	20	60	0.018389	6.73E-06	10	31	0.008304	6.71E-06
	t_2	5	15	0.006929	6.08E-06	21	63	0.015709	6.68E-06	11	33	0.00886	6.60E-06
	t_3	4	12	0.005407	5.62E-06	22	66	0.016738	7.98E-06	11	34	0.008164	7.34E-06
	t_4	12	37	0.015989	5.43E-06	23	69	0.016536	7.96E-06	12	37	0.010165	8.76E-06
	t_5	12	37	0.017033	5.43E-06	56	169	0.041178	8.47E-06	12	37	0.009695	8.76E-06
10,000	t_1	5	15	0.009567	4.92E-06	20	60	0.028081	9.52E-06	10	31	0.013467	9.49E-06
	t_2	5	15	0.011175	8.60E-06	21	63	0.03296	9.45E-06	11	33	0.015336	9.33E-06
	t_3	4	12	0.009999	7.95E-06	23	69	0.032998	5.64E-06	12	36	0.016367	5.19E-06
	t_4	12	37	0.024755	7.67E-06	24	72	0.031536	5.63E-06	13	39	0.016389	6.19E-06
	t_5	12	37	0.023768	7.67E-06	57	172	0.073391	8.98E-06	13	39	0.017317	6.19E-06
50,000	t_1	5	16	0.035628	2.20E-06	22	66	0.098167	5.32E-06	11	34	0.054976	5.30E-06
	t_2	5	16	0.03746	3.85E-06	23	69	0.11354	5.28E-06	12	36	0.055763	5.22E-06
	t_3	4	13	0.035312	3.56E-06	24	72	0.1112	6.31E-06	12	37	0.055787	5.80E-06
	t_4	13	40	0.088376	3.98E-06	25	75	0.11938	6.29E-06	13	40	0.06102	6.93E-06
	t_5	13	40	0.08748	3.98E-06	60	181	0.27048	8.47E-06	13	40	0.064139	6.93E-06
100,000	t_1	5	16	0.070098	3.11E-06	22	66	0.24879	7.52E-06	11	34	0.094677	7.50E-06
	t_2	5	16	0.066656	5.44E-06	23	69	0.23271	7.47E-06	12	36	0.099045	7.38E-06
	t_3	4	13	0.059075	5.03E-06	24	72	0.20778	8.92E-06	12	37	0.10176	8.21E-06
	t_4	13	40	0.16086	5.63E-06	25	75	0.21683	8.90E-06	13	40	0.11361	9.79E-06
	t_5	13	40	0.15224	5.63E-06	61	184	0.57751	8.99E-06	13	40	0.11484	9.79E-06

TABLE 4 Numerical results of Problem 4

		Algorithm 1				PRPFR				HCGP			
DIM	INP	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM
1,000	t_1	1	4	0.001591	0	19	57	0.011641	5.45E-06	9	28	0.002226	9.77E-06
	t_2	1	4	0.000927	0	19	57	0.003844	9.85E-06	10	30	0.002698	7.64E-06
	t_3	7	23	0.002857	8.75E-06	20	60	0.004185	8.87E-06	9	28	0.002166	6.32E-06
	t_4	1	7	0.001103	0	50	151	0.008999	9.99E-06	30	91	0.005137	7.60E-06
	t_5	1	9	0.002018	0	110	332	0.024385	9.64E-06	1	5	0.001174	0
5,000	t_1	1	4	0.002033	0	20	60	0.010725	6.10E-06	10	31	0.006579	5.46E-06
	t_2	1	4	0.001814	0	21	63	0.011173	5.50E-06	10	31	0.005721	8.55E-06
	t_3	8	25	0.008563	7.05E-06	21	63	0.011246	9.91E-06	10	30	0.005524	7.07E-06
	t_4	1	7	0.002675	0	53	160	0.025946	9.42E-06	31	95	0.016139	8.85E-06
	t_5	1	9	0.003019	0	116	350	0.050032	9.68E-06	1	5	0.001374	0
10,000	t_1	1	4	0.00256	0	20	60	0.017556	8.62E-06	10	31	0.010383	7.72E-06
	t_2	1	4	0.002921	0	21	63	0.026692	7.78E-06	11	33	0.009643	6.04E-06
	t_3	8	25	0.011035	9.97E-06	22	66	0.019025	7.01E-06	10	30	0.009142	9.99E-06
	t_4	1	7	0.003905	0	54	163	0.047946	9.99E-06	32	97	0.027089	9.39E-06
	t_5	1	9	0.004428	0	119	359	0.09649	9.17E-06	1	5	0.002023	0
50,000	t_1	1	4	0.007052	0	21	63	0.071295	9.64E-06	11	33	0.031648	8.63E-06
	t_2	1	4	0.007597	0	22	66	0.066349	8.70E-06	11	34	0.031757	6.76E-06
	t_3	9	28	0.04068	5.17E-06	23	69	0.067232	7.84E-06	11	33	0.032884	5.59E-06
	t_4	1	7	0.012622	0	57	172	0.15778	9.43E-06	34	103	0.089874	8.20E-06
	t_5	1	9	0.019196	0	125	377	0.37734	9.20E-06	1	5	0.006158	0
100,000	t_1	1	4	0.012959	0	22	66	0.13804	6.82E-06	11	34	0.063684	6.10E-06
	t_2	1	4	0.01309	0	23	69	0.13403	6.15E-06	11	34	0.0632	9.56E-06
	t_3	9	28	0.072451	7.31E-06	24	72	0.13884	5.54E-06	11	33	0.061035	7.90E-06
	t_4	1	7	0.020594	0	58	175	0.33469	1.00E-05	34	104	0.1824	9.66E-06
	t_5	1	9	0.027575	0	127	383	0.75897	9.96E-06	1	5	0.010724	0

TABLE 5 Numerical results of Problem 5

		Algorithm 1				PRPFR				HCGP			
DIM	INP	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM
1,000	t_1	7	21	0.004388	9.88E-06	23	69	0.027045	9.86E-06	12	36	0.004099	9.85E-06
	t_2	8	24	0.003152	2.57E-06	23	69	0.006047	9.48E-06	12	36	0.00383	9.47E-06
	t_3	8	24	0.00325	3.70E-06	23	69	0.007138	8.35E-06	12	36	0.003794	8.34E-06
	t_4	8	24	0.003724	3.87E-06	22	66	0.00705	9.17E-06	11	34	0.003555	9.17E-06
	t_5	7	21	0.003472	9.32E-06	22	66	0.007609	5.41E-06	11	34	0.003727	5.40E-06
5,000	t_1	7	21	0.007928	2.75E-06	25	75	0.026502	5.52E-06	13	39	0.011194	5.52E-06
	t_2	7	21	0.010504	3.03E-06	25	75	0.028712	5.31E-06	13	39	0.010396	5.31E-06
	t_3	7	21	0.0107	3.68E-06	24	72	0.021278	9.35E-06	12	37	0.011246	9.35E-06
	t_4	7	21	0.010014	3.72E-06	24	72	0.023806	5.13E-06	12	37	0.010799	5.13E-06
	t_5	6	18	0.009371	8.94E-06	23	69	0.022786	6.05E-06	12	36	0.012104	6.05E-06
10,000	t_1	6	18	0.018051	7.86E-06	25	75	0.0437	7.80E-06	13	39	0.021837	7.80E-06
	t_2	6	18	0.016816	7.67E-06	25	75	0.04265	7.50E-06	13	39	0.022963	7.50E-06
	t_3	6	18	0.016091	7.04E-06	25	75	0.045588	6.61E-06	13	39	0.021916	6.61E-06
	t_4	6	18	0.016917	4.46E-06	24	72	0.043416	7.26E-06	12	37	0.022192	7.26E-06
	t_5	6	18	0.015769	2.85E-06	23	69	0.039561	8.56E-06	12	36	0.02134	8.56E-06
50,000	t_1	6	19	0.068204	2.86E-06	26	78	0.18787	8.72E-06	13	40	0.078189	8.72E-06
	t_2	6	19	0.056087	2.80E-06	26	78	0.18437	8.39E-06	13	40	0.077414	8.39E-06
	t_3	6	19	0.055964	2.59E-06	26	78	0.19127	7.39E-06	13	40	0.078046	7.39E-06
	t_4	6	18	0.054176	5.59E-06	25	75	0.19165	8.12E-06	13	39	0.081114	8.12E-06
	t_5	6	18	0.053036	3.30E-06	24	72	0.14784	9.57E-06	12	37	0.071817	9.57E-06
100,000	t_1	6	19	0.1076	3.41E-06	27	81	0.34293	6.17E-06	14	42	0.16723	6.17E-06
	t_2	6	19	0.10816	3.29E-06	27	81	0.364	5.93E-06	14	42	0.17723	5.93E-06
	t_3	6	19	0.12679	2.90E-06	27	81	0.37623	5.23E-06	14	42	0.16716	5.23E-06
	t_4	6	18	0.11378	7.89E-06	26	78	0.46513	5.74E-06	13	40	0.15761	5.74E-06
	t_5	6	18	0.12326	4.65E-06	25	75	0.58247	6.77E-06	13	39	0.16082	6.77E-06

TABLE 6 Numerical results of Problem 6

		Algorithm 1				PRPFR				HCGP			
DIM	INP	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM
1,000	t_1	7	23	0.003133	5.95E-06	24	73	0.072008	6.78E-06	7	21	0.002167	8.45E-06
	t_2	7	23	0.002896	5.56E-06	23	70	0.008086	9.18E-06	7	21	0.002509	2.20E-06
	t_3	6	20	0.003037	2.08E-06	18	55	0.007074	7.53E-06	4	13	0.001261	1.26E-10
	t_4	7	23	0.002774	6.03E-06	25	76	0.009862	7.09E-06	6	18	0.002118	1.98E-06
	t_5	8	25	0.003588	3.73E-06	24	72	0.010133	9.60E-06	5	14	0.001823	1.18E-06
5,000	t_1	8	26	0.008867	2.00E-06	25	76	0.032226	8.07E-06	8	25	0.0063	2.17E-06
	t_2	8	26	0.009042	1.87E-06	25	76	0.028052	5.81E-06	7	21	0.006013	4.92E-06
	t_3	6	20	0.007878	4.66E-06	19	58	0.019263	8.96E-06	4	13	0.003715	2.82E-10
	t_4	8	26	0.009993	2.03E-06	26	79	0.043618	8.44E-06	6	18	0.004745	4.43E-06
	t_5	8	25	0.010185	8.34E-06	26	78	0.030357	6.08E-06	5	14	0.004772	2.65E-06
10,000	t_1	8	26	0.016016	2.83E-06	26	79	0.042	6.07E-06	8	25	0.011125	3.07E-06
	t_2	8	26	0.017106	2.64E-06	25	76	0.040285	8.21E-06	7	21	0.00995	6.95E-06
	t_3	6	20	0.012398	6.59E-06	20	61	0.035545	6.74E-06	4	13	0.006491	4.00E-10
	t_4	8	26	0.01691	2.87E-06	27	82	0.049436	6.35E-06	6	18	0.00751	6.26E-06
	t_5	9	28	0.018777	1.77E-06	26	78	0.041621	8.59E-06	5	14	0.00674	3.75E-06
50,000	t_1	8	26	0.057205	6.32E-06	27	82	0.15397	7.22E-06	8	25	0.0408	6.86E-06
	t_2	8	26	0.055839	5.91E-06	26	79	0.15613	9.77E-06	8	25	0.03903	1.34E-06
	t_3	7	23	0.048831	2.22E-06	21	64	0.12179	8.02E-06	4	13	0.022145	8.93E-10
	t_4	8	26	0.055615	6.41E-06	28	85	0.12957	7.55E-06	7	22	0.035054	2.58E-08
	t_5	9	28	0.061647	3.96E-06	28	84	0.30012	5.44E-06	5	14	0.023888	8.38E-06
100,000	t_1	8	26	0.10488	8.94E-06	28	85	0.24969	5.43E-06	8	25	0.071697	9.70E-06
	t_2	8	26	0.11113	8.35E-06	27	82	0.22995	7.35E-06	8	25	0.075784	1.89E-06
	t_3	7	23	0.093533	3.13E-06	22	67	0.20493	6.03E-06	4	13	0.041847	1.26E-09
	t_4	8	26	0.10972	9.06E-06	29	88	0.25431	5.68E-06	7	22	0.068693	3.65E-08
	t_5	9	28	0.13371	5.60E-06	28	84	0.24616	7.69E-06	6	18	0.056298	8.28E-08

TABLE 7 Numerical results of Problem 7

		Algorithm 1				PRPFR				HCGP			
DIM	INP	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM
1,000	t_1	7	25	0.002586	3.31E-06	12	37	0.018339	9.04E-06	2	8	0.001002	0
	t_2	7	25	0.002806	2.00E-06	12	37	0.00194	5.47E-06	2	8	0.000774	0
	t_3	7	25	0.002201	1.91E-06	12	37	0.002232	5.22E-06	2	8	0.00078	0
	t_4	8	28	0.002308	1.68E-06	14	43	0.002235	3.51E-06	2	8	0.001154	7.02E-15
	t_5	8	28	0.00263	2.41E-06	14	43	0.002575	5.04E-06	2	8	0.001136	1.05E-14
5,000	t_1	7	25	0.004778	7.40E-06	13	40	0.005633	5.92E-06	2	8	0.001737	0
	t_2	7	25	0.015085	4.48E-06	13	40	0.005312	3.59E-06	2	8	0.001384	0
	t_3	7	25	0.006713	4.27E-06	13	40	0.005278	3.42E-06	2	8	0.00138	1.57E-14
	t_4	8	28	0.006006	3.75E-06	14	43	0.005893	7.84E-06	2	8	0.001685	1.57E-14
	t_5	8	28	0.004982	5.39E-06	15	46	0.006995	3.30E-06	2	8	0.001389	2.36E-14
10,000	t_1	8	28	0.009993	1.17E-06	13	40	0.010623	8.37E-06	2	8	0.00258	2.22E-14
	t_2	7	25	0.009458	6.34E-06	13	40	0.010733	5.07E-06	2	8	0.002645	2.22E-14
	t_3	7	25	0.008096	6.04E-06	13	40	0.009986	4.84E-06	2	8	0.002599	0
	t_4	8	28	0.010372	5.31E-06	15	46	0.011647	3.25E-06	2	8	0.002581	2.22E-14
	t_5	8	28	0.01173	7.62E-06	15	46	0.012334	4.66E-06	2	8	0.003249	6.66E-14
50,000	t_1	8	28	0.037214	2.62E-06	14	43	0.036045	5.48E-06	2	8	0.008043	4.97E-14
	t_2	8	28	0.036128	1.59E-06	14	43	0.045307	3.32E-06	2	8	0.00811	0
	t_3	8	28	0.030424	1.52E-06	14	43	0.039362	3.17E-06	2	8	0.007281	0
	t_4	9	31	0.035098	1.33E-06	15	46	0.042763	7.26E-06	2	8	0.008416	1.49E-13
	t_5	9	31	0.036143	1.91E-06	16	49	0.044959	3.05E-06	2	8	0.007379	2.48E-13
100,000	t_1	8	28	0.060661	3.71E-06	14	43	0.075845	7.75E-06	2	8	0.01501	7.02E-14
	t_2	8	28	0.061047	2.25E-06	14	43	0.073335	4.70E-06	2	8	0.01457	1.40E-13
	t_3	8	28	0.064175	2.14E-06	14	43	0.075028	4.48E-06	2	8	0.013776	1.40E-13
	t_4	9	31	0.066658	1.88E-06	16	49	0.09746	3.01E-06	2	8	0.015092	2.11E-13
	t_5	9	31	0.065785	2.70E-06	16	49	0.087173	4.32E-06	2	8	0.014057	4.92E-13

TABLE 8 Numerical results of Problem 8

		Algorithm 1				PRPFR				HCGP			
DIM	INP	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM
1,000	t_1	50	156	0.016899	3.69E-06	26	80	0.021806	6.94E-06	22	68	0.004837	8.39E-06
	t_2	50	156	0.014829	3.64E-06	29	89	0.006164	8.35E-06	24	75	0.005822	9.42E-06
	t_3	36	114	0.011215	5.15E-06	22	68	0.004758	7.13E-06	17	54	0.004126	9.23E-06
	t_4	40	126	0.01108	9.15E-06	41	125	0.008258	8.09E-06	27	83	0.005879	7.76E-06
	t_5	37	117	0.012109	6.82E-06	41	125	0.010434	8.64E-06	26	80	0.00586	7.93E-06
5,000	t_1	51	159	0.14978	5.07E-06	27	83	0.022232	6.61E-06	28	86	0.018328	8.60E-06
	t_2	49	153	0.047529	4.07E-06	29	89	0.022802	9.74E-06	28	86	0.022628	8.63E-06
	t_3	34	108	0.033381	3.70E-06	24	74	0.018578	8.40E-06	15	47	0.011866	5.96E-06
	t_4	63	195	0.067442	3.66E-06	38	116	0.028043	8.63E-06	28	86	0.018113	9.02E-06
	t_5	36	114	0.038464	6.86E-06	41	125	0.032816	9.65E-06	27	84	0.019408	8.94E-06
10,000	t_1	54	168	0.092577	5.80E-06	26	80	0.038902	8.05E-06	26	81	0.03268	8.68E-06
	t_2	48	150	0.090042	8.89E-06	29	89	0.036467	8.36E-06	27	83	0.03493	8.85E-06
	t_3	39	123	0.072636	5.53E-06	23	71	0.029877	9.61E-06	23	72	0.030303	9.37E-06
	t_4	41	129	0.07106	3.32E-06	35	107	0.045569	8.90E-06	28	87	0.033426	9.63E-06
	t_5	36	114	0.066589	6.83E-06	40	122	0.05392	9.07E-06	28	87	0.033969	9.06E-06
50,000	t_1	50	156	0.29213	6.22E-06	33	101	0.1517	9.45E-06	28	86	0.12371	8.64E-06
	t_2	49	153	0.28355	4.00E-06	29	89	0.14824	9.81E-06	27	83	0.11988	8.97E-06
	t_3	28	90	0.1875	9.44E-06	26	80	0.12233	9.70E-06	29	89	0.12048	8.56E-06
	t_4	71	219	0.42194	3.80E-06	35	107	0.15478	9.47E-06	25	78	0.11194	8.87E-06
	t_5	36	114	0.20871	6.82E-06	40	122	0.17869	9.87E-06	25	78	0.11239	9.51E-06
100,000	t_1	46	144	0.58371	5.72E-06	33	101	0.31899	7.79E-06	21	66	0.19114	8.49E-06
	t_2	46	144	0.55133	5.40E-06	29	89	0.29049	8.65E-06	28	86	0.2486	8.54E-06
	t_3	41	129	0.51383	3.63E-06	25	77	0.24196	8.74E-06	27	83	0.2424	9.54E-06
	t_4	53	165	0.67444	3.79E-06	31	95	0.29581	8.66E-06	26	81	0.23408	9.87E-06
	t_5	37	117	0.50434	6.86E-06	39	119	0.38842	7.91E-06	30	92	0.26463	8.12E-06

TABLE 9 Numerical results of Problem 9

		Algorithm 1				PRPFR				HCGP			
DIM	INP	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM
1,000	t_1	1	11	0.002721	0	15	47	0.089234	5.81E-06	5	18	0.002078	6.19E-06
	t_2	1	11	0.001746	0	16	50	0.007191	3.94E-06	1	6	0.001262	0
	t_3	1	2	0.001072	0	16	50	0.004405	7.20E-06	1	2	0.001656	0
	t_4	1	2	0.00113	0	42	129	0.009504	8.46E-06	1	2	0.00121	0
	t_5	1	4	0.001603	0	92	280	0.024416	9.43E-06	1	4	0.001795	0
5,000	t_1	1	11	0.005022	0	16	50	0.013498	4.87E-06	6	20	0.005543	5.19E-06
	t_2	1	11	0.005464	0	16	50	0.014526	8.82E-06	1	6	0.002429	0
	t_3	1	2	0.002663	0	17	53	0.01721	6.04E-06	1	2	0.001721	0
	t_4	1	2	0.002943	0	44	135	0.037461	8.94E-06	1	2	0.002223	0
	t_5	1	4	0.003532	0	97	295	0.08065	9.02E-06	1	4	0.002225	0
10,000	t_1	1	11	0.009011	0	16	50	0.026324	6.89E-06	6	20	0.011232	7.35E-06
	t_2	1	11	0.008905	0	17	53	0.026495	4.68E-06	1	6	0.003434	0
	t_3	1	2	0.004139	0	17	53	0.025465	8.54E-06	1	2	0.002688	0
	t_4	1	2	0.004124	0	45	138	0.06664	8.70E-06	1	2	0.002999	0
	t_5	1	4	0.004682	0	99	301	0.13289	9.08E-06	1	4	0.003071	0
50,000	t_1	1	11	0.030857	0	17	53	0.090062	5.78E-06	6	21	0.037906	2.74E-06
	t_2	1	11	0.030101	0	18	56	0.095829	3.92E-06	1	6	0.012283	0
	t_3	1	2	0.013804	0	18	56	0.10114	7.16E-06	1	2	0.00854	0
	t_4	1	2	0.014521	0	47	144	0.25129	9.19E-06	1	2	0.010251	0
	t_5	1	4	0.020523	0	104	316	0.52998	8.68E-06	1	4	0.011815	0
100,000	t_1	1	11	0.052496	0	17	53	0.16535	8.17E-06	6	21	0.070365	3.87E-06
	t_2	1	11	0.052904	0	18	56	0.17929	5.55E-06	1	6	0.021476	0
	t_3	1	2	0.02428	0	19	59	0.19395	3.80E-06	1	2	0.016473	0
	t_4	1	2	0.026506	0	48	147	0.45947	8.94E-06	1	2	0.016398	0
	t_5	1	4	0.031785	0	106	322	1.0038	8.74E-06	1	4	0.026064	0

TABLE 10 Numerical results of Problem 10

		Algorithm 1				PRPFR				HCGP			
DIM	INP	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM
1,000	t_1	6	26	0.003245	4.09E-06	18	56	0.021142	7.65E-06	7	24	0.002247	7.25E-06
	t_2	6	25	0.002901	3.10E-06	17	53	0.003727	5.16E-06	8	27	0.002146	3.64E-06
	t_3	3	15	0.002057	5.30E-06	17	53	0.003747	6.86E-06	8	27	0.002273	6.59E-06
	t_4	32	107	0.0094	8.09E-06	41	126	0.008161	8.03E-06	29	91	0.006529	8.00E-06
	t_5	56	181	0.016383	9.81E-06	95	289	0.018192	8.54E-06	58	178	0.010835	9.16E-06
5,000	t_1	6	26	0.010773	9.14E-06	19	59	0.013422	6.92E-06	8	26	0.005877	6.56E-06
	t_2	6	25	0.007682	6.92E-06	18	56	0.013232	4.67E-06	8	27	0.005788	8.13E-06
	t_3	6	24	0.008519	2.72E-06	18	56	0.012685	6.21E-06	9	29	0.006356	5.97E-06
	t_4	33	111	0.038927	9.91E-06	43	132	0.034899	8.85E-06	30	94	0.024955	9.90E-06
	t_5	59	190	0.051698	9.20E-06	100	304	0.072257	8.53E-06	61	187	0.039821	9.60E-06
10,000	t_1	7	28	0.018929	2.60E-06	19	59	0.029302	9.79E-06	8	26	0.013684	9.28E-06
	t_2	6	25	0.018031	9.79E-06	18	56	0.027955	6.60E-06	9	29	0.015667	4.65E-06
	t_3	6	24	0.017828	3.84E-06	18	56	0.029693	8.78E-06	9	29	0.015803	8.44E-06
	t_4	34	113	0.073849	9.43E-06	44	135	0.060644	8.80E-06	31	96	0.045654	9.83E-06
	t_5	60	193	0.1138	9.74E-06	102	310	0.13406	8.74E-06	62	191	0.084273	9.62E-06
50,000	t_1	7	28	0.0685	5.81E-06	20	62	0.1053	8.86E-06	8	27	0.048481	5.49E-06
	t_2	6	26	0.06236	4.49E-06	19	59	0.10122	5.97E-06	9	30	0.051215	2.75E-06
	t_3	6	24	0.060073	8.60E-06	19	59	0.10167	7.95E-06	9	30	0.051292	4.99E-06
	t_4	36	119	0.26072	7.78E-06	46	141	0.23082	9.70E-06	32	100	0.18823	9.59E-06
	t_5	63	202	0.47618	9.14E-06	107	325	0.55535	8.73E-06	66	202	0.34695	8.58E-06
100,000	t_1	7	28	0.13674	8.22E-06	21	65	0.23232	5.07E-06	8	27	0.19058	7.76E-06
	t_2	6	26	0.1239	6.35E-06	19	59	0.2143	8.45E-06	9	30	0.12354	3.89E-06
	t_3	7	27	0.12969	2.39E-06	20	62	0.22516	4.55E-06	9	30	0.11039	7.06E-06
	t_4	36	120	0.54909	9.93E-06	47	144	0.52804	9.64E-06	33	102	0.39391	9.53E-06
	t_5	64	205	0.98074	9.68E-06	109	331	1.2373	8.94E-06	67	205	0.84191	9.43E-06

- Control parameters: We select $\rho = 0.8$, $\sigma = 10^{-4}$, $\gamma = 1.2$. As for PRPFR and HCGP, all parameters are as selected in Zhou et al[37] and Koorapetse and Kaelo,[38] respectively.
- Stopping condition: Iterations are stopped when $\|G^{(k)}\| \leq 10^{-5}$ and/or the number of iterations exceed 1,000 without reaching a solution.
- The symbol – is used to indicate that an algorithm failed due to number of iterations exceeding 1,000 with reaching a solution.

We consider the following test problems where the operator G is $G(t) = (g_1(t), g_2(t), \cdots, g_n(t))^T$ and $t = (t_1, t_2, \cdots, t_n)^T$.
Problem 1. The Exponential Function.[39]

$$\begin{aligned} &g_1(t) = e^{t_1} - 1, \\ &g_i(t) = e^{t_i} + t_i - 1, \text{ for } i = 2, 3, \cdots, n, \text{ and } B = R^n_+. \end{aligned}$$

Problem 2. Modified Logarithmic Function.[39]

$$\begin{aligned} &g_i(t) = \ln(t_i + 1) - \frac{t_i}{n}, \text{ for } i = 1, 2, \cdots, n, \\ &\text{and } B = \{t \in R^n : \sum_{i=1}^{n} t_i \leq n, t_i > -1, i = 1, 2, \cdots, n\}. \end{aligned}$$

Problem 3. Nonsmooth Function I.[40]

$$\begin{aligned} &g_i(t) = 2t_i - \sin |t_i|, i = 1, 2, 3, \cdots, n, \\ &\text{and } B = \{t \in R^n : \sum_{i=1}^{n} t_i \leq n, t_i \geq 0, i = 1, 2, \cdots, n\}. \end{aligned}$$

It is clear that Problem 3 is nonsmooth at $t = 0$.
Problem 4. Strictly Convex Function I.[39]

$$g_i(t) = e^{t_i} - 1, \text{ for } i = 1, 2, \cdots, n, \text{ and } B = R^n_+.$$

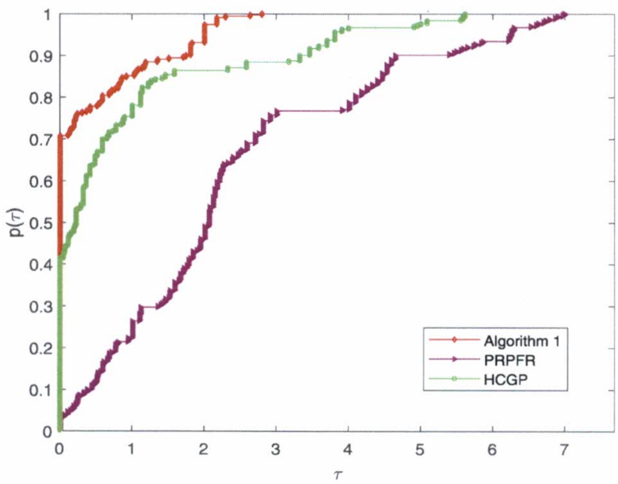

FIGURE 1 Performance profiles for the number of iterations [Colour figure can be viewed at wileyonlinelibrary.com]

Problem 5.[41] Tridiagonal Exponential Function.

$$g_1(t) = t_1 - e^{\cos(h(t_1+t_2))},$$
$$g_i(t) = t_i - e^{\cos(h(t_{i-1}+t_i+t_{i+1}))}, \text{ for } i = 2, \cdots, n-1,$$
$$g_n(t) = t_n - e^{\cos(h(t_{n-1}+t_n))},$$
$$h = \frac{1}{n+1} \text{ and } B = R^n_+.$$

Problem 6. Nonsmooth Function II.[42]

$$g_i(t) = t_i - \sin|t_i - 1|, \; i = 1, 2, 3, \cdots, n.$$
$$\text{and } B = \{t \in R^n : \sum_{i=1}^{n} t_i \leq n, t_i \geq -1, i = 1, 2, \cdots, n\}.$$

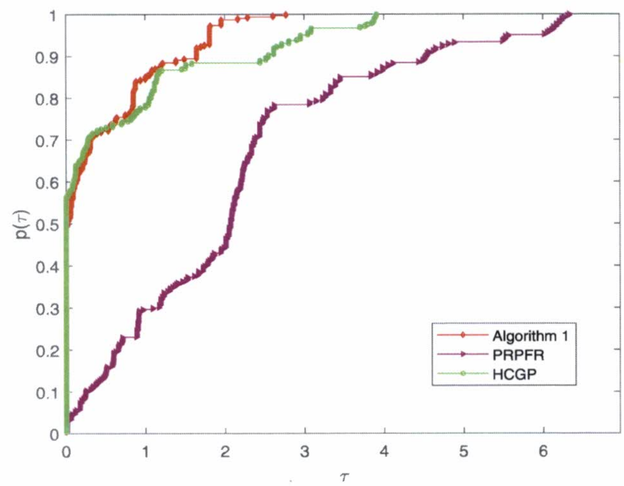

FIGURE 2 Performance profiles for the number of function evaluations [Colour figure can be viewed at wileyonlinelibrary.com]

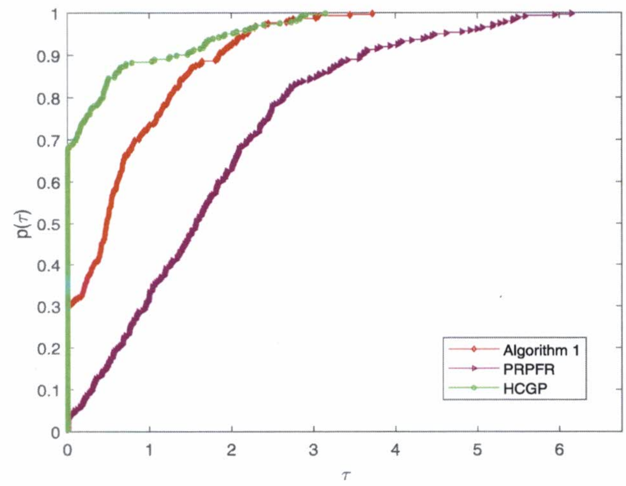

FIGURE 3 Performance profiles for the CPU time (in seconds) [Colour figure can be viewed at wileyonlinelibrary.com]

Problem 7. Penalty Function 1.[43]

$$v_i = \sum_{i=1}^{n} t_i^2, \; c = 10^{-5}$$
$$g_i(t) = 2c(t_i - 1) + 4(v_i - 0.25)t_i, \; i = 1, 2, 3, \cdots, n.$$
$$\text{and } B = R^n_+.$$

Problem 8. Pursuit-Evasion problem.[17]

$$g_i(t) = \sqrt{8}t_i - 1, \; i = 1, 2, 3, \cdots, n.$$
$$\text{and } B = R^n_+.$$

Problem 9.[40]

$$g_1(t) = 2t_1 + \sin t_1 - 1,$$
$$g_i(t) = -t_{i-1} + 2t_i + \sin t_i - 1, \text{ for } i = 2, \cdots, n-1,$$
$$g_n(t) = 2t_n + \sin t_n - 1,$$
$$\text{and } B = R^n_+.$$

Problem 10.[44]

$$g_i(t) = e^{t_i^2} + 3 \sin t_1 \cos t_i - 1, \text{for } i = 1, 2, \cdots, n$$
$$\text{and } B = R^n_+.$$

Results of the numerical experiments are presented in Tables 1–10. From the tables, it can be seen that all the algorithms successfully solved all the problems. However, to ascertain which is more efficient, the Dolan and Morè[45] performance profile is employed. Figures 1–3 were plotted based on the profile, and each plot corresponds to one of the standard metrics used. In Figure 1 which is base on ITER, Algorithm 1 is the most efficient having the least number of iteration in 70% of the problems as against PRPFR and HCGP the least in 42% and 5%, respectively. Base on FVAL, in Figure 2, there was a competition between Algorithm 1 and HCGP with each having the least number of function evaluation in over 50% of the problems as against HCGP with the least in just 5%. Moreover, based on TIME, in Figure 3, HCGP is the most efficient having the least CPU time in almost 70% of the problems as against Algorithm 1 and PRPFR having the least in 30% and 5%, respectively. This may be because HCGP has less computation cost than Algorithm 1 and PRPFR. In summary, we can conclude that Algorithm 1 is more efficient than PRPFR and HCGP in terms of ITER, competes with HCGP in terms of FVAL and second most efficient in terms of TIME.

5 | APPLICATION IN SIGNAL PROCESSING

5.1 | General description

As mentioned in Section 1, it is interesting to note that many problems can be converted into monotone nonlinear equations of the form (1). In particular, we consider the reconstruction of an original signal, say $\tilde{t} \in \mathbb{R}^n$, by minimizing an objective function that contains a linear least square error term and a sparseness-including ℓ_1 regularization term

$$\min_t \tau \|t\|_1 + \frac{1}{2}\|Bt - b\|_2^2, \tag{34}$$

where $t \in \mathbb{R}^n$, $b \in \mathbb{R}^k$ is an observation, $B \in \mathbb{R}^{k\times n}(k << n)$ is a linear operator, τ is a non-negative parameter, $\|t\|_2$ is an Euclidean norm of t, and $\|t\|_1 := \sum_{i=1}^{n} |t_i|$ is the ℓ_1-norm of t. Iterative methods for solving problem (34) has received much attention by many researchers in the literature (see previous studies[46–52]). The most popular method among these methods is the gradient projection method for sparse reconstruction (GPRS) proposed by Figueiredo et al.[53] Other gradient-based iterative algorithms include the coordinate gradient descent proposed by Yun and Toh[54] and the accelerated gradient projection method proposed in Loris et al[55] among others. These methods enjoyed good performance, even though they required the knowledge of the gradient. The first step of the GPRS method is to express (34) as a quadratic problem using the following process.

Consider a point $t \in \mathbb{R}^n$ such that $t = u - v$, $u, v \geq 0$; u and v are chosen in such a way that t is splitted into its positive and negative parts such that for all $i = 1, 2, \cdots, n$, $u_i = (t_i)_+$, $v = (-t_i)_+$ and $(.)_+ = \max\{0, .\}$. By definition of ℓ_1-norm, we have $\|t\|_1 = e_n^T u + e_n^T v$, where $e_n = (1, 1, \cdots, 1)^T \in \mathbb{R}^n$. Hence, the ℓ_1-norm problem (34) can be transformed into

$$\min_{u,v} \frac{1}{2}\|b - B(u - v)\|_2^2 + \tau e_n^T u + \tau e_n^T v, \text{ such that } u \geq 0, v \geq 0. \tag{35}$$

However, from Figueiredo et al,[53] problem (35) can also be rewritten as the quadratic program problem with box constraints

$$\min_{z} \frac{1}{2} z^T D z + c^T z, \text{ such that } z \geq 0, \tag{36}$$

where $z = \begin{bmatrix} u \\ v \end{bmatrix}$, $c = \tau e_{2n} + \begin{bmatrix} -y \\ y \end{bmatrix}$, $y = B^T b$, and $D = \begin{bmatrix} B^T B & -B^T B \\ -B^T B & B^T B \end{bmatrix}$.

Clearly, D is a positive semi-definite matrix, which implies that problem (36) is a convex quadratic problem.

Xiao and Zhu[56] translated problem (36) into a linear variational inequality problem which is equivalent to a linear complementary problem. Furthermore, they pointed out that z is a solution of the linear complementary problem if and only if it is a solution of the nonlinear equation:

$$F(z) = \min\{z, Dz + c\} = 0, \tag{37}$$

where $F(z)$ is continuous and monotone; see Pang[57] and Xiao et al.[58] Hence, solving problem (34) is equivalent to solving problem (1). Therefore, Algorithm 1 can be applied to solve problem (34) effectively.

5.2 | Numerical experiment

In this subsection, our focus is utilizing Algorithm 1 in the restoration of one dimensional sparse signal. In this experiment, we consider a simple compressive sensing possible situation, where our aim is to reconstruct a one-dimensional sparse signal of length n from k observations. The quality of restoration is measured by using their mean squared error (MSE) to the original signal $\tilde{t}$ defined by

$$MSE := \frac{1}{n}\|\tilde{t} - t^*\|^2,$$

where t^* is a recovered signal. However, we choose the parameters $\rho = 0.8$, $\beta = 1.0$, and $\sigma = 10^{-4}$ for Algorithm 1.

The signal size is chosen as $n = 2^{11}$, $k = 2^9$, and the original signal contains 2^6 randomly non-zero elements. Furthermore, during the experiments, a randomly Gaussian matrix B using the Matlab command *randn(k, n)* is generated. In

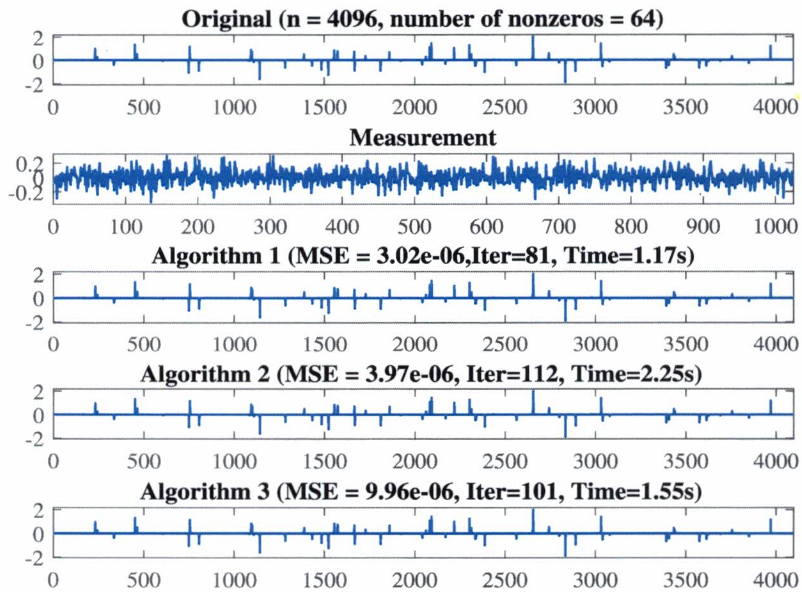

FIGURE 4 Reconstruction of sparse signal. From top to bottom is the original signal, the measurement, and the reconstructed signals by Algorithms 1–3 [Colour figure can be viewed at wileyonlinelibrary.com]

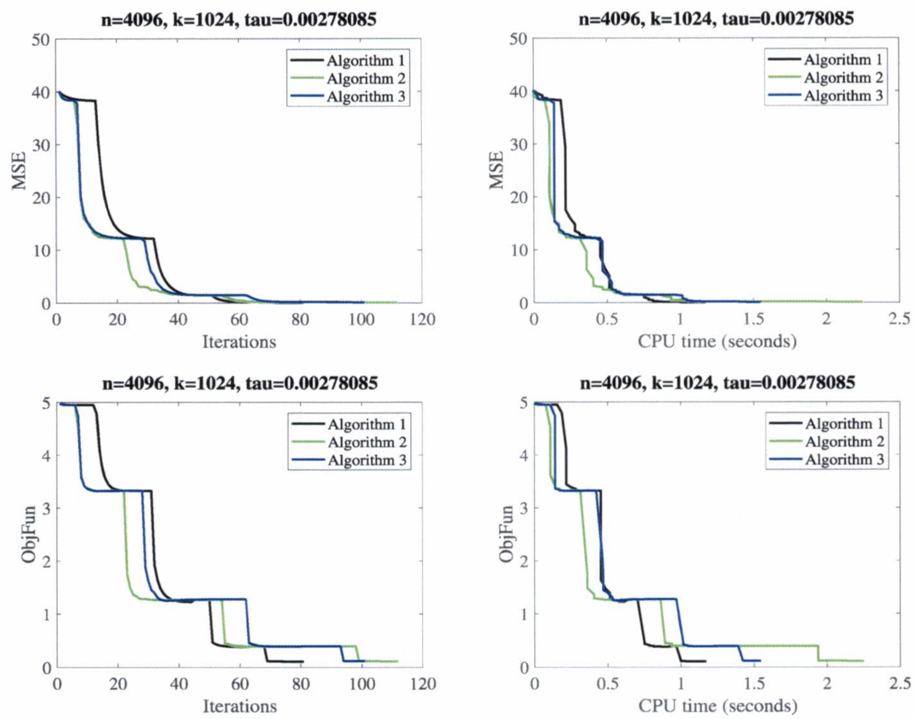

FIGURE 5 Comparison results of Algorithms 1–3. The x axis represents the number of iterations (top and bottom left) and the CPU time in seconds (top and bottom right); the y axis represent the MSE (top left and top right) and the function values (bottom left and right) [Colour figure can be viewed at wileyonlinelibrary.com]

the test, the measurement b is computed as $b = B\tilde{t} + \delta$, where δ is the Gaussian noise which is normally distributed with mean 0 and variance 10^{-4}.

To evaluate the performance of Algorithm 1, we test it against similar algorithms which were specifically designed to solve monotone nonlinear equations with convex constraints and reconstruction of sparse signal in compressive sensing. The compared algorithms include CGD method[56] denoted as Algorithm 2 and PCG method[59] denoted as Algorithm 3. For fairness in comparing the algorithms, the iteration process of all the algorithms started at $t^{(0)} := B^T b$ and terminates when

$$\frac{|f_k - f^{(k-1)}|}{|f^{(k-1)}|} < 10^{-5},$$

where $f_k := \frac{1}{2}\|b - Bt^{(k)}\|_2^2 + \tau\|t^{(k)}\|_1$ is the objective function.

In Figure 4, we provide the numerical results consisting of the original sparse signal, the measurement and the reconstructed signal by Algorithms 1–3, respectively. In addition, in Figure 5, we give a visual illustration of the performance of each algorithm relative to their convergence behavior from the view of merit function values and relative error as the

TABLE 11 Twelve experimental results on signal recovery

	Algorithm 1			Algorithm 2			Algorithm 3		
	CPU	ITER	MSE	CPU	ITER	MSE	CPU	ITER	MSE
	1.38	85.00	3.39E-06	1.91	114.00	4.51E-06	1.81	111.00	3.79E-06
	1.34	86.00	4.42E-06	1.69	102.00	5.82E-06	1.69	109.00	4.77E-06
	1.20	82.00	3.43E-06	1.95	129.00	3.77E-06	1.70	111.00	3.60E-06
	1.39	85.00	3.20E-06	1.84	126.00	4.77E-06	1.66	110.00	2.76E-06
	1.17	81.00	3.02E-06	2.25	112.00	3.97E-06	1.55	101.00	9.96E-06
	1.22	84.00	2.60E-06	1.86	125.00	3.19E-06	2.05	94.00	5.12E-06
	1.11	74.00	3.92E-06	1.84	122.00	6.86E-06	1.70	115.00	4.00E-06
	1.31	84.00	1.41E-06	1.72	109.00	1.93E-06	1.52	104.00	2.65E-06
	1.30	84.00	3.99E-06	1.78	121.00	3.11E-05	1.72	114.00	4.19E-06
	1.72	83.00	3.17E-06	1.91	125.00	3.65E-06	1.59	100.00	3.38E-06
	1.27	79.00	3.24E-06	1.66	112.00	3.75E-06	1.64	106.00	3.45E-06
	1.45	88.00	1.86E-06	2.16	141.00	2.24E-06	1.58	104.00	2.00E-06
Average	1.32	82.92	3.1375E-06	1.88	119.83	6.29667E-06	1.68	106.58	4.13917E-06

number of iterations and computing time increases. Comparing the four algorithms, in Figure 5, it is not difficult to see that the original signal is recovered by all the four algorithms. However, Algorithm 1 proves to be more efficient having lesser number of iterations, computing time and most importantly lesser MSE. To further illustrate the efficiency of Algorithm 1, we repeated the experiment on twelve different noise samples. Each time the experiment is run, Algorithm 1 prove to be more efficient than Algorithms 2 and 3 in almost all the experiments in terms of the number of iterations, CPU time, and MSE. See summary in Table 11.

6 | CONCLUSIONS

In this paper, we proposed a hybrid CG algorithm that combine the HS and DY methods for solving monotone operator equations. This was achieved by taking the convex combination of the HS and DY CG parameters using an appropriate hybridization parameter. The corresponding direction satisfies the descent condition if the operator is the gradient of a function to be minimized. In addition, the direction is bounded independent of the line search. We also proved the global convergence of the hybrid method under some nice and achievable assumptions. The efficiency of the algorithm was shown by comparing its numerical performance with existing algorithms using some benchmark test problems. Finally, application of the algorithm in signal recovery was given.

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CONFLICT OF INTEREST

This work does not have any conflict of interest.

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