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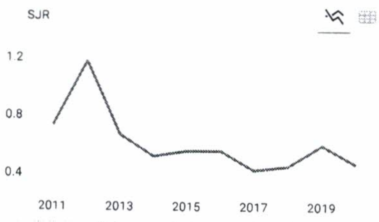

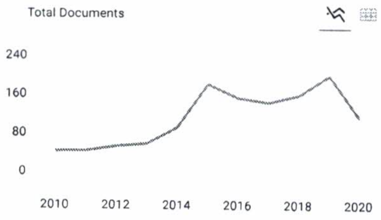

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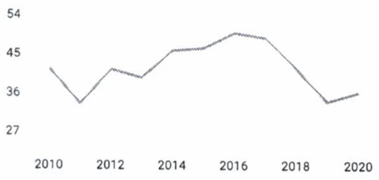

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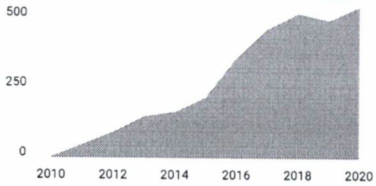

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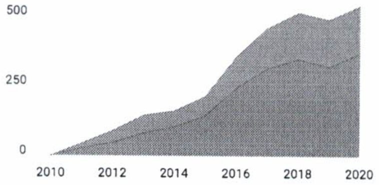

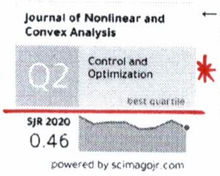

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ON A SEQUENCE OF QUASI-NONEXPANSIVE MAPPINGS IN A GEODESIC SPACE WITH CURVATURE BOUNDED ABOVE

NAWARAT EKKARNTRONG*, NUTTAPOL PAKKARANANG†, BANCHA PANYANAK‡,
AND PONGSAKORN YOTKAEW

This work is dedicated to the memory of Professor Wataru Takahashi.

ABSTRACT. The purpose of this paper is to present Δ-convergence and strong convergence theorems for quasi-nonexpansive sequences in the setting of a geodesic space with curvature bounded above by one. The results can be applied to the image recovery problem for a countable family of closed convex subsets of such spaces and also applied to the optimization problem for convex functions.

1. Introduction

One of the most important problems in optimization theory is the problem of finding a minimizer of a convex function f on a space X into $(-\infty,\infty]$, i.e., find $x \in X$ such that $f(x) = \min_{y\in X} f(y)$. The set of all minimizers of f is denoted by $\operatorname{argmin}_X f$. Let λ be a positive real number and f a proper lower semicontinuous convex function of a complete CAT(0) space X into $(-\infty,\infty]$. It is known that the resolvent $J_{\lambda f}$ of λf given by

$$J_{\lambda f}x := \operatorname*{argmin}_{y\in X}\left\{f(y) + \frac{1}{2\lambda}d^2(y,x)\right\} \tag{1.1}$$

for $x \in X$ is a well-defined nonexpansive mapping of X into itself such that the set $\operatorname{Fix}(J_{\lambda f})$ of all fixed points of $J_{\lambda f}$ coincides with $\operatorname{argmin}_X f$. In other word, $z = J_{\lambda f}z$ where $\lambda > 0$ if and only if $z \in \operatorname{argmin}_X f$; see [2, 9, 23] for more details. There have been considerably many interesting results of iterative methods for approximating minimizers of the function f. One of the most successful methods is the proximal point algorithm introduced by Martinet [22] and studied more generally by Rockafellar [26], which is defined by $x_1 \in X$ and

$$x_{n+1} = J_{\lambda_n f}x_n$$

for $n \in \mathbb{N}$, where $\{\lambda_n\}$ is a sequence of positive real numbers. By using this algorithm, Bačák [1] showed that the sequence $\{x_n\}$ is Δ-convergent to a minimizer of f if $\operatorname{argmin}_X f \neq \varnothing$ and $\sum_{n=1}^{\infty}\lambda_n = \infty$. Later, Kimura and Kohsaka [14] obtained

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‡Corresponding author.

Δ-convergence and strong convergence theorems for two variant of the proximal point algorithm.

Ohta and Pálfia [25] showed that the resolvent $J_{\lambda f}$ given by (1.1) is still well defined and the proximal point algorithm is Δ-convergent to a minimizer of f in a complete CAT(1) space X such that $\operatorname{diam} X < \pi/2$. However, it is known that the direct application of the techniques in CAT(0) spaces to CAT(1) spaces does not work. For example, the resolvent $J_{\lambda f}$ is not necessary to be nonexpansive in CAT(1) spaces.

On the other hand, Kimura and Kohsaka [13] introduced another type of resolvents of convex functions in an admissible complete CAT(1) space X and showed that if f is a proper lower semicontinuous convex function of X into $(-\infty, \infty]$, then the resolvent R_f of f given by

$$R_f(x) := \operatorname*{argmin}_{y\in X} \{f(y) + \tan d(y,x) \sin d(y,x)\} \tag{1.2}$$

for $x \in X$ is a well-defined and single-valued mapping of X into itself such that $\operatorname{Fix}(R_f) = \operatorname{argmin}_X f$. In this case, R_f is quasi-nonexpansive, i.e., $d(R_f x, z) \le d(x,z)$ for all $x \in X$ and for all $z \in \operatorname{Fix}(R_f)$. They [15] and Espínola and Nicolae [6] also obtained a Δ-convergence theorem for the proximal point algorithm in CAT(1) spaces. Recently, Kimura and Kohsaka [16] obtained the following convergence theorem for the modified proximal point algorithm in CAT(1) spaces.

Theorem 1.1 ([16, Theorems 5.1 (ii) and 5.2]). *Let X be an admissible complete CAT(1) space, f a proper lower semicontinuous convex function of X into $(-\infty, \infty]$ with a minimizer, $R_{\eta f}$ the resolvent of ηf for all $\eta > 0$, and $\{x_n\}$ a sequence defined by $x_1, u \in X$ and*

$$x_{n+1} = \alpha_n u \oplus (1-\alpha_n) R_{\lambda_n f} x_n$$

for $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $[0,1]$ and $\{\lambda_n\}$ is a sequence of positive real numbers such that $\lim_n \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n^2 = \infty$, and $\liminf_n \lambda_n > 0$. If either $\{\alpha_n\} \subset [0,1)$ or $\lim_n \lambda_n = \infty$, then the sequence $\{x_n\}$ converges strongly to the point Pu, where P denotes the metric projection of X onto $\operatorname{argmin}_X f$.

Motivated by the previous works, we present Δ-convergence and strong convergence theorems for quasi-nonexpansive sequences in the setting of an admissible complete CAT(1) space which extend some results of [19, 20]. Then we give some their applications to optimization problems for convex functions which extend Theorem 1.1 and some results of [1, 6, 14–16]. Moreover, the results can be applied to the image recovery problem for a countable family of closed convex subsets of an admissible complete CAT(1) space.

2. Preliminaries

Let X be a geodesic space. A *geodesic triangle* $\Delta(u,v,w)$ consists of three points $u, v, w \in X$ and all the images of each geodesic part joining two of them. For a triangle $\Delta(u,v,w)$ in X satisfying $d(u,v)+d(v,w)+d(w,u) < 2\pi$, we can find the comparison triangle $\Delta(\bar{u},\bar{v},\bar{w})$ in the unit sphere $\mathbb{S}^2$ in $\mathbb{R}^3$; that is, each corresponding edge has the same length as that of original triangle. If for any $p, q \in \Delta(u,v,w)$ and their corresponding comparison points $\bar{p}, \bar{q} \in \Delta(\bar{u},\bar{v},\bar{w})$, the inequality $d(p,q) \le d_{\mathbb{S}^2}(\bar{p},\bar{q})$

holds, then we call X a *CAT(1) space*. See [3–5, 7, 10, 11] for more details on CAT(1) spaces.

Let (X, d) be a CAT(1) space. Given a point $t \in [0, 1]$ and two points $v, w \in X$ such that $d(v, w) < \pi$, we use the notation $tv \oplus (1 - t)w$ for a unique point u in the unique geodesic segment $[v, w]$ such that

$$d(u, v) = (1 - t)d(v, w) \text{ and } d(u, w) = td(v, w).$$

A subset C of X is called *convex* if $tv \oplus (1 - t)w \in C$ for all $v, w \in C$ such that $d(v, w) < \pi$. Recall that a CAT(1) space is *admissible* if $d(v, w) < \pi/2$ for all $v, w \in X$ and the diameter of X is denoted by diam X.

The following lemmas are essentially needed for our main results.

Lemma 2.1 ([17, Corollary 2.2]). *Let $t \in [0, 1]$ and u, v, w be three points in a* CAT(1) *space (X, d) such that $d(u, v) + d(v, w) + d(w, u) < 2\pi$. Then*

$$\begin{aligned}&\cos d(tv \oplus (1 - t)w, u) \sin d(v, w)\\ &\geq \cos d(v, u) \sin(td(v, w)) + \cos d(w, u) \sin((1 - t)d(v, w)).\end{aligned}$$

Let C be a closed convex subset of a complete CAT(1) space (X, d) such that $d(v, C) := \inf_{w \in C} d(v, w) < \pi/2$ for all $v \in X$. Then the metric projection P_C from X onto C is well defined; that is, for each $v \in X$, there exists the unique point $P_C v \in C$ satisfying

$$d(v, P_C v) = \inf_{w \in C} d(v, w).$$

Lemma 2.2 ([28, Proposition 2.3]). *Let X be an admissible complete CAT(1) space, C a nonempty closed convex subset of X, $x \in X$ and $z \in C$. Then $z = P_C x$ if and only if $\cos d(x, y) \leq \cos d(x, z) \cos d(y, z)$ for all $y \in C$.*

Recall that a bounded sequence $\{x_n\}$ in a metric space X is *Δ-convergent to* $x \in X$ [12, 21] if $\{z \in X : \limsup_k d(z, x_{n_k}) = \inf_{y \in X} \limsup_k d(y, x_{n_k})\} = \{x\}$ for all subsequences $\{x_{n_k}\}$ of $\{x_n\}$. In this case, the element x is called *the Δ-limit* of $\{x_n\}$. It is known that if $\{x_n\}$ is a *spherically bounded sequence* in a CAT(1) space X, that is,

$$\inf_{x \in X} \limsup_{n \to \infty} d(x, x_n) < \pi/2,$$

then $\{x_n\}$ has a Δ-convergent subsequence; see [5, Corollary 4.4].

Lemma 2.3. *Let X be an admissible complete CAT(1) space, C a nonempty closed convex subset of X, and $\{x_n\}$ be a sequence in X. Suppose that $\{x_n\}$ is a Fejér sequence with respect to C, that is,*

$$d(x_{n+1}, y) \leq d(x_n, y) \text{ for all } n \in \mathbb{N} \text{ and for all } y \in C.$$

Then $\{P_C x_n\}$ is a Cauchy sequence. Moreover, if $\{x_n\}$ is Δ-convergent to $x \in C$ and $\{P_C x_n\}$ is strongly convergent to z, then $x = z$.

Proof. Let $y \in C$ and $n, k \in \mathbb{N}$. By the assumption, we have $d(x_n, P_C x_n) \leq d(x_n, y) \leq d(x_1, y) < \pi/2$. It follows from Lemma 2.2 that

$$\begin{aligned}\cos d(x_{n+k}, P_C x_{n+k}) &\geq \cos d(x_{n+k}, P_C x_{n+k}) \cos d(P_C x_n, P_C x_{n+k})\\ &\geq \cos d(x_{n+k}, P_C x_n)\\ &\geq \cos d(x_n, P_C x_n).\end{aligned}$$

As a consequence, we have $d(x_{n+k}, P_C x_{n+k}) \leq d(x_n, P_C x_n) \leq d(x_1, y) < \pi/2$ and

$$\cos d(P_C x_n, P_C x_{n+k}) \geq \frac{\cos d(x_n, P_C x_n)}{\cos d(x_{n+k}, P_C x_{n+k})}.$$

It then follows that $\alpha := \lim_n d(x_n, P_C x_n)$ exists which is less than $\pi/2$. Moreover, we have

$$\begin{aligned} 2\sin^2 \frac{d(P_C x_n, P_C x_{n+k})}{2} &= 1 - \cos d(P_C x_n, P_C x_{n+k}) \\ &\leq 1 - \frac{\cos d(x_n, P_C x_n)}{\cos d(x_{n+k}, P_C x_{n+k})} \\ &\leq 1 - \frac{\cos d(x_n, P_C x_n)}{\cos \alpha} \\ &= \frac{\cos \alpha - \cos d(x_n, P_C x_n)}{\cos \alpha}. \end{aligned}$$

By elementary trigonometry, it is not hard to see that $\{P_C x_n\}$ is a Cauchy sequence.

Now, let z be the limit point of $\{P_C x_n\}$ and $x \in C$ be the Δ-limit point of $\{x_n\}$. Note that $d(x_n, P_C x_n) \leq d(x_n, x)$ for all $n \in \mathbb{N}$. Then we have that following estimate:

$$\begin{aligned} \limsup_{n\to\infty} d(x_n, x) &\leq \limsup_{n\to\infty} d(x_n, z) \\ &\leq \limsup_{n\to\infty} (d(x_n, P_C x_n) + d(P_C x_n, z)) \\ &\leq \limsup_{n\to\infty} d(x_n, x). \end{aligned}$$

By the uniqueness of an asymptotic center of $\{x_n\}$, we obtain that $x = z$ and the proof is finished. □

Proposition 2.4 ([20, Proposition 3.1]). *Let $\{x_n\}$ be a spherically bounded sequence of a complete* CAT(1) *space X. Suppose that $\lim_n d(x_n, z)$ exists for all $z \in \omega_\Delta(\{x_n\})$, where $\omega_\Delta(\{x_n\})$ is the set of all Δ-limit points of $\{x_n\}$. Then $\{x_n\}$ Δ-converges to an element of $\omega_\Delta(\{x_n\})$.*

Lemma 2.5 ([8, Proposition 2.3]). *Let X be a complete CAT(1) space, $p \in X$, and $\{x_n\}$ be a sequence in X such that $\limsup_n d(x_n, p) < \pi/2$. If $\{x_n\}$ is Δ-convergent to $x \in X$, then*

$$d(x, p) \leq \liminf_{n\to\infty} d(x_n, p).$$

Lemma 2.6 ([19, Lemma 3.1]). *Let X be an admissible CAT(1) space, $u, v, w \in X$ and $t \in [0, 1]$. Then*

$$\begin{aligned} &2\sin^2 \frac{d(tv \oplus (1-t)w, u)}{2} \\ &\leq (1-\gamma) 2\sin^2 \frac{d(w, u)}{2} + \gamma \left(1 - \frac{\cos d(v, u)}{\sin d(v, w) \tan \frac{t}{2} d(v, w) + \cos d(v, w)}\right), \end{aligned}$$

where

$$\gamma = \begin{cases} 1 - \dfrac{\sin((1-t) d(v, w))}{\sin d(v, w)} & \text{if } v \neq w, \\ t & \text{otherwise.} \end{cases}$$

Lemma 2.7 ([13, Lemma 2.3]). *Let u, v, w be three points in a* CAT(1) *space (X, d) such that $d(v, u) \leq \pi/2$ and $d(w, u) \leq \pi/2$, and let $t \in [0, 1]$. Then*

$$\cos d(tv \oplus (1-t)w, u) \geq t \cos d(v, u) + (1-t) \cos d(w, u).$$

Moreover, we have $d(tv \oplus (1-t)w, u) \leq \max\{d(v, u), d(w, u)\}$ [29, Lemma 3.4].

Lemma 2.8 ([27]). *Let $\{a_n\}$, $\{c_n\}$ be sequences of nonnegative real numbers, $\{b_n\}$ be a sequence of real numbers, and $\{\alpha_n\}$ be a sequence in $[0, 1]$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$. Suppose that*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n b_n + c_n \quad \text{for all } n \in \mathbb{N}.$$

If $\limsup_k b_{m_k} \leq 0$ for every subsequence $\{a_{m_k}\}$ of $\{a_n\}$ satisfying

$$\liminf_{k\to\infty}(a_{m_k+1} - a_{m_k}) \geq 0,$$

then $\lim_n a_n = 0$.

3. Convergence theorems

Throughout the rest of this paper, we do assume that X is an admissible complete CAT(1) space. A sequence $\{T_n\}$ of mappings on X is said to:

- satisfy *Condition (B)* [24] if every Δ-limit of Δ-convergent subsequence of $\{x_n\}$ belongs to $\bigcap_{n=1}^{\infty} \operatorname{Fix}(T_n)$ whenever $\{x_n\}$ is a sequence in X such that $\lim_n d(x_n, T_n x_n) = 0$.
- be *strongly quasi-nonexpansive* if $\{T_n\}$ is a sequence of quasi-nonexpansive mappings and $\lim_n d(x_n, T_n x_n) = 0$ whenever $p \in \bigcap_{n=1}^{\infty} \operatorname{Fix}(T_n)$ and $\{x_n\}$ is a sequence of X such that $\sup_{n\in\mathbb{N}} d(x_n, p) < \pi/2$ and $\lim_n \cos d(x_n, p)/\cos d(T_n x_n, p) = 1$.

Note that every nonexpansive mapping T on X is Δ*-demiclosed*, that is, the constant sequence $\{T\}$ satisfies Condition (B). We also know that the metric projection P_C from X onto a closed convex subset C of X is quasi-nonexpansive and Δ-demiclosed. Moreover, the constant sequence $\{P_C\}$ is strongly quasi-nonexpansive; see [19].

We first prove the following Δ-convergence theorem.

Theorem 3.1. *Let $\{T_n\}$ be a sequence of quasi-nonexpansive mappings of X into itself such that $F := \bigcap_{n=1}^{\infty} \operatorname{Fix}(T_n) \neq \emptyset$. Define a sequence $\{x_n\}$ in X by $x_1 \in X$ and*

$$x_{n+1} = T_n x_n \quad \text{for } n \in \mathbb{N}.$$

If $\{T_n\}$ is a strongly quasi-nonexpansive sequence satisfying Condition (B), then the sequence $\{x_n\}$ is Δ-convergent to an element x of $\bigcap_{n=1}^{\infty} \operatorname{Fix}(T_n)$. Moreover, $\{P_F x_n\}$ is strongly convergent to x.

Proof. Note that F is closed and convex. Let $p \in F$. For each $n \in \mathbb{N}$, we have

$$d(x_{n+1}, p) = d(Tx_n, p) \leq d(x_n, p).$$

This implies that $\sup_{n\in\mathbb{N}} d(x_n, p) < \pi/2$, $\lim_n d(x_n, p)$ exists and

$$\lim_{n\to\infty}(d(x_n, p) - d(T_n x_n, p)) = 0.$$

Consequently, we obtain

$$\lim_{n\to\infty} \frac{\cos d(x_n, p)}{\cos d(T_n x_n, p)} = 1.$$

Using the strong quasi-nonexpansiveness of $\{T_n\}$ gives

$$\lim_{n\to\infty} d(x_n, T_n x_n) = 0.$$

It then follows from the condition (B) of $\{T_n\}$ that $\omega_\Delta(\{x_n\}) \subset F$. As a consequence of Lemma 2.4 that $\{x_n\}$ Δ-converges to an element $x \in F$. Note that $\{x_n\}$ is a Fejér sequence with respect to F. Hence we can conclude from Lemma 2.3 that $\{P_F x_n\}$ is strongly convergent to x, and the proof is finished. □

The following result is supplement to Theorem 3.8 of [20].

Theorem 3.2. *Let $\{T_n\}$ be a sequence of quasi-nonexpansive mappings of X into itself such that $F := \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n) \neq \varnothing$. Define a sequence $\{x_n\}$ in X by $x_1 \in X$ and*

$$x_{n+1} = \alpha_n x_n \oplus (1 - \alpha_n) T_n x_n \quad \text{for } n \in \mathbb{N},$$

where $\{\alpha_n\}$ is sequences in $[a, b] \subset (0, 1)$. If $\{T_n\}$ satisfies Condition (B), then the sequence $\{x_n\}$ is Δ-convergent to an element $p \in F$. Moreover, $\{P_F x_n\}$ is strongly convergent to p.

Proof. As an immediately consequence of Theorem 3.1 and the following lemma, we obtain the result. □

Lemma 3.3. *Let $\{T_n\}$ be a sequence of quasi-nonexpansive mappings of X into itself such that $F := \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n) \neq \varnothing$. Define a sequence $\{S_n\}$ of mappings on X by*

$$S_n := \alpha_n I \oplus (1 - \alpha_n) T_n \quad \text{for } n \in \mathbb{N}$$

where $\{\alpha_n\} \subset [a, b] \subset (0, 1)$. Then the following statements hold.

(i) *$\{S_n\}$ is a strongly quasi-nonexpansive sequence.*
(ii) *If $\{T_n\}$ satisfies Condition (B), then so does $\{S_n\}$.*

Proof. (i) For each $n \in \mathbb{N}$, it is not hard to see that $\mathrm{Fix}(S_n) = \mathrm{Fix}(T_n)$. Let $x \in X$ and $p \in \mathrm{Fix}(S_n)$. By Lemma 2.7, we have

$$d(S_n x, p) \le \max\{d(x_n, p), d(T_n x, p)\} = d(x, p),$$

that is, S_n is quasi-nonexpansive. Let $\{x_n\}$ be a sequence in X and $p \in F$ such that $\sup_{n\in\mathbb{N}} d(x_n, p) < \pi/2$ and $\lim_n \cos d(x_n, p)/\cos d(S_n x_n, p) = 1$. Put $t := \limsup_n d(x_n, T_n x_n)$. Note that $\sup_{n\in\mathbb{N}} d(S_n x_n, p) < \pi/2$. Then there exists a subsequence $\{n_k\}$ of $\{n\}$ such that $\lim_k d(x_{n_k}, T_{n_k} x_{n_k}) = t$, $\lim_k d(S_{n_k} x_{n_k}, p) = \lim_k d(x_{n_k}, p) = s < \pi/2$ and $\lim_k \alpha_{n_k} = \alpha \in (0, 1)$. It follows from Lemma 2.1 and the quasi-nonexpansiveness of T_{n_k} that

$$\begin{aligned} &\cos d(S_{n_k} x_{n_k}, p) \sin d(x_{n_k}, T_{n_k} x_{n_k}) \\ &\ge \cos d(x_{n_k}, p)(\sin(\alpha_{n_k} d(x_{n_k}, T_{n_k} x_{n_k})) + \sin((1 - \alpha_{n_k}) d(x_{n_k}, T_{n_k} x_{n_k}))) \end{aligned}$$

for all $k \in \mathbb{N}$. Letting $k \to \infty$ yields

$$\cos s \sin t \ge \cos s(\sin \alpha t + \sin(1 - \alpha)t).$$

As a consequence, we obtain that $t = 0$. Hence, we have

$$\lim_{n\to\infty} d(x_n, S_n x_n) = \lim_{n\to\infty} \alpha_n d(x_n, T_n x_n) = 0.$$

Therefore, $\{S_n\}$ is strongly quasi-nonexpansive.

(ii) Assume that $\{T_n\}$ satisfies Condition (B). Let $\{x_n\}$ be a sequence in X and $x \in X$ such that $\lim_n d(x_n, S_n x_n) = 0$ and suppose that $\{x_n\}$ is Δ-convergent to x. Since $\{\alpha_n\} \subset [a, b] \subset (0, 1)$, we have $\lim_n d(x_n, T_n x_n) = \lim_n (d(x_n, S_n x_n)/\alpha_n) = 0$. It follows from Condition (B) of $\{T_n\}$ that $x \in F$ and this completes the proof. □

We next prove the following strong convergence theorem.

Theorem 3.4. *Let T_n be a sequence of quasi-nonexpansive mappings of X into itself such that $F := \bigcap_{n=1}^{\infty} \operatorname{Fix}(T_n) \neq \varnothing$. Define a sequence $\{x_n\}$ in X by $x_1, u \in X$ and*

$$x_{n+1} = \beta_n x_n \oplus (1 - \beta_n)\left(\alpha_n u \oplus (1 - \alpha_n) T_n x_n\right) \quad \textit{for } n \in \mathbb{N},$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$ such that $\lim_n \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\limsup_n \beta_n < 1$. Suppose that $\{T_n\}$ satisfies Condition (B), and either $\liminf_n \beta_n > 0$ or $\{T_n\}$ is strongly quasi-nonexpansive. If either diam $X < \pi/2$ *or $\sum_{n=1}^{\infty} \alpha_n^2 = \infty$, then the sequence $\{x_n\}$ converges strongly to the point $p = P_F u$.*

Proof. Note that F is closed and convex. For each $n \in \mathbb{N}$, $y_n := \alpha_n u \oplus (1 - \alpha_n) T_n x_n$ and $p := P_F u$. It follows from Lemma 2.6 and the quasi-nonexpansiveness of T_n that

$$\begin{aligned} 2\sin^2 \frac{d(y_n, p)}{2} &\leq (1 - \gamma_n) 2 \sin^2 \frac{d(x_n, p)}{2} \\ &\quad + \gamma_n \left(1 - \frac{\cos d(u, p)}{\sin d(u, T_n x_n) \tan \frac{\alpha_n}{2} d(u, T_n x_n) + \cos d(u, T_n x_n)}\right), \end{aligned}$$

where

$$\gamma_n := \begin{cases} 1 - \dfrac{\sin((1 - \alpha_n) d(u, T_n x_n))}{\sin d(u, T_n x_n)} & \text{if } u \neq T_n x_n, \\ \alpha_n & \text{otherwise.} \end{cases}$$

Using this together with Lemma 2.7, we have the following estimate:

$$\begin{aligned} &2\sin^2 \frac{d(x_{n+1}, p)}{2} \\ &\leq \beta_n 2 \sin^2 \frac{d(x_n, p)}{2} + (1 - \beta_n) 2 \sin^2 \frac{d(y_n, p)}{2} \\ &\leq (1 - (1 - \beta_n)\gamma_n) 2 \sin^2 \frac{d(x_n, p)}{2} \\ &\quad + (1 - \beta_n)\gamma_n \left(1 - \frac{\cos d(u, p)}{\sin d(u, T_n x_n) \tan \frac{\alpha_n}{2} d(u, T_n x_n) + \cos d(u, T_n x_n)}\right). \end{aligned}$$

This implies that

$$a_{n+1} \leq (1 - (1 - \beta_n)\gamma_n) a_n + (1 - \beta_n)\gamma_n b_n$$

for all $n \in \mathbb{N}$, where

$$\begin{aligned} a_n &:= 2\sin^2 \frac{d(x_n, p)}{2}; \\ b_n &:= 1 - \frac{\cos d(u,p)}{\sin d(u, T_n x_n) \tan \frac{\alpha_n}{2} d(u, T_n x_n) + \cos d(u, T_n x_n)}. \end{aligned}$$

Since either $\operatorname{diam} X < \pi/2$ or $\sum_{n=1}^{\infty} \alpha_n^2 = \infty$, we obtain $\sum_{n=1}^{\infty}(1-\beta_n)\gamma_n = \infty$.

To apply Lemma 2.8, it suffices to show that $\limsup_k b_{m_k} \leq 0$ for every subsequence $\{a_{m_k}\}$ of $\{a_n\}$ satisfying

$$\liminf_{k\to\infty}(a_{m_k+1} - a_{m_k}) \geq 0.$$

Let $\{a_{m_k}\}$ be a subsequence of $\{a_n\}$ such that

$$\liminf_{k\to\infty}(a_{m_k+1} - a_{m_k}) \geq 0.$$

Using this together with Lemma 2.7, we have

$$\begin{aligned} 0 &\leq \liminf_{k\to\infty} (a_{m_k+1} - a_{m_k}) \\ &= \liminf_{k\to\infty} (\cos d(x_{m_k}, p) - \cos d(x_{m_{k+1}}, p)) \\ &\leq \liminf_{k\to\infty} (\cos d(x_{m_k}, p) - (\beta_{m_k} \cos d(x_{m_k}, p) + (1-\beta_{m_k}) \cos d(y_{m_k}, p))) \\ &= \liminf_{k\to\infty}(1-\beta_{m_k})(\cos d(x_{m_k}, p) - \cos d(y_{m_k}, p)) \\ &= \liminf_{k\to\infty}(1-\beta_{m_k})(\cos d(x_{m_k}, p) - \cos d(T_{m_k} x_{m_k}, p)) \\ &\leq \limsup_{k\to\infty}(1-\beta_{m_k})(\cos d(x_{m_k}, p) - \cos d(T_{m_k} x_{m_k}, p)) \\ &\leq 0. \end{aligned}$$

It follows from $\limsup_n \beta_n < 1$ that

$$\lim_{k\to\infty} (\cos d(x_{m_k}, p) - \cos d(T_{m_k} x_{m_k}, p)) = 0.$$

By Lemma 2.7, we have $d(T_n x_n, p) \leq d(x_n, p) \leq \max\{d(x_1, p), d(u, p)\} < \pi/2$ for all $n \in \mathbb{N}$. Consequently, we obtain

$$\lim_{k\to\infty} \cos d(x_{m_k}, p)/\cos d(T_{m_k} x_{m_k}, p) = 1.$$

If $\{T_n\}$ is strongly quasi-nonexpansive, then

$$\lim_{k\to\infty} d(x_{m_k}, T_{m_k} x_{m_k}) = 0.$$

On the other hand, assume that $\liminf_n \beta_n > 0$. Passing to a suitable subsequence still denoted by $\{m_k\}$, we assume that

$$A := \lim_{k\to\infty} d(x_{m_k}, T_{m_k} x_{m_k}) \text{ and } \beta := \lim_{k\to\infty} \beta_{m_k} \in (0,1).$$

Note that $\lim_k d(y_{m_k}, T_{m_k}x_{m_k}) = 0$ and $\lim_k \cos d(x_{m_k}, p)/\cos d(x_{m_{k+1}}, p) = 1$. Therefore, by Lemma 2.1, we obtain

$$\begin{aligned}
\sin A &= \lim_{k\to\infty} \sin d(x_{m_k}, T_{m_k}x_{m_k}) \\
&= \lim_{k\to\infty} \frac{\cos d(x_{m_{k+1}}, p)}{\cos d(x_{m_k}, p)} \sin d(x_{m_k}, y_{m_k}) \\
&\geq \lim_{k\to\infty} \left(\sin \beta_{m_k} d(x_{m_k}, y_{m_k}) + \frac{\cos d(y_{m_k}, p)}{\cos d(x_{m_k}, p)} \sin(1-\beta_{m_k}) d(x_{m_k}, y_{m_k}) \right) \\
&= \sin \beta A + \sin(1-\beta)A.
\end{aligned}$$

By elementary trigonometry, we have $A = 0$, that is,

$$\lim_{k\to\infty} d(x_{m_k}, T_{m_k}x_{m_k}) = 0.$$

Let $\{z_k\}$ be a Δ-convergent subsequence of $\{x_{m_k}\}$ such that

$$\lim_{k\to\infty} d(z_k, u) = \lim_{k\to\infty} d(x_{m_k}, u).$$

Define a sequence $\{w_n\}$ in X by

$$\{w_n\} := \begin{cases} w_{m_k} = x_{m_k} & \text{for all } k \in \mathbb{N}, \\ w_n = p & \text{otherwise.} \end{cases}$$

Obviously, $\lim_n d(w_n, T_n w_n) = 0$. It follows from Condition (B) of $\{T_n\}$ that the Δ-limit z of $\{z_k\}$ belongs to F. By Lemma 2.5, the definitions of the Δ-limit, and $p = P_F u$, we obtain

$$\liminf_{k\to\infty} d(T_{m_k}x_{m_k}, u) = \liminf_{k\to\infty} d(x_{m_k}, u) = \lim_{k\to\infty} d(z_k, u) \geq d(z, u) \geq d(p, u).$$

Therefore, we have

$$\begin{aligned}
&\limsup_{k\to\infty} b_{m_k} \\
&= \limsup_{k\to\infty} \left(1 - \frac{\cos d(u,p)}{\sin d(u, T_{m_k}x_{m_k}) \tan \frac{\alpha_{m_k}}{2} d(u, T_{m_k}x_{m_k}) + \cos d(u, T_{m_k}x_{m_k})} \right) \\
&= \limsup_{k\to\infty} \left(1 - \frac{\cos d(u,p)}{\cos d(u, T_{m_k}x_{m_k})} \right) \\
&\leq 0.
\end{aligned}$$

As a result, Lemma 2.8 guarantees the strong convergence of $\{x_n\}$ to p, and this completes the proof. □

Corollary 3.5. *Let $\{T_n\}$ be a sequence of quasi-nonexpansive mappings of X into itself such that $F := \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n) \neq \varnothing$. Define a sequence $\{x_n\}$ in X by $x_1, u \in X$ and*

$$x_{n+1} = \beta_n x_n \oplus (1-\beta_n)(\alpha_n u \oplus (1-\alpha_n) T_n x_n) \quad \text{for } n \in \mathbb{N},$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ such that $\lim_n \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $0 < \liminf_n \beta_n \leq \limsup_n \beta_n < 1$. Suppose, in addition, that $\{T_n\}$ satisfies Condition (B). If either $\operatorname{diam} X < \pi/2$ or $\sum_{n=1}^{\infty} \alpha_n^2 = \infty$, then the sequence $\{x_n\}$ converges strongly to the point $P_F u$.

As an immediately consequence of Theorem 3.4, we obtain the following result which extends Theorem 3.2 of [19].

Corollary 3.6. *Let $\{T_n\}$ be a sequence of quasi-nonexpansive mappings of X into itself such that $F := \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n) \neq \varnothing$. Define a sequence $\{x_n\}$ in X by $x_1, u \in X$ and*

$$x_{n+1} = \alpha_n u \oplus (1-\alpha_n)T_n x_n \quad \text{for } n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$ such that $\lim_n \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Suppose, in addition, that $\{T_n\}$ is a strongly quasi-nonexpansive sequence satisfying Condition (B). If either $\operatorname{diam} X < \pi/2$ or $\sum_{n=1}^{\infty} \alpha_n^2 = \infty$, then the sequence $\{x_n\}$ converges strongly to the point $P_F u$.

4. Applications

4.1. **Application to the optimization problem.** Recall that a function $f : X \to (-\infty, \infty]$ is *proper* if $D(f) := \{v \in X : f(v) \in \mathbb{R}\} \neq \varnothing$. It is said to be *convex* if $f(\alpha v \oplus (1-\alpha)w) \leq \alpha f(v) + (1-\alpha)f(w)$ for all $v, w \in X, \alpha \in [0,1]$. It is also said to be *lower semicontinuous* (*Δ-lower semicontinuous*, respectively) if $f(v) \leq \liminf_n f(v_n)$ whenever $\{v_n\} \subset X$ such that $\{v_n\}$ is strongly convergent to $v \in D(f)$ ($\{v_n\}$ is Δ-convergent to $v \in D(f)$, respectively).

We know that if f is a proper, convex and lower semi-continuous function, then it is Δ-lower semicontinuous (see [14]). Moreover, for each $\lambda > 0$, the set $F(R_{\lambda f})$ of fixed points of the resolvent $R_{\lambda f}$ coincides with the set $\operatorname{argmin}_X f$ of minimizers of f.

Lemma 4.1 ([15, Lemma 3.1]). *Let $\lambda > 0$ and $f : X \to (-\infty, \infty]$ be a proper convex and lower semi-continuous function, and $R_{\lambda f}$ be the resolvent of λf. Then*

$$\cos d(R_{\lambda f}x, x) \cos d(R_{\lambda f}x, z) \geq \cos d(x, z)$$

holds for all $x \in X$ and $z \in \operatorname{argmin}_X f$.

Lemma 4.2 ([16, Lemma 3.1 (i)]). *Let $f : X \to (-\infty, \infty]$ be a proper, convex and lower semi-continuous function such that $\operatorname{argmin}_X f \neq \varnothing$, and $\{\lambda_n\}$ a sequence of positive real numbers. If $\liminf_n \lambda_n > 0$, that $\{R_{\lambda_n f}\}$ satisfies Condition (B).*

Remark 4.3 (see also [16, Lemma 3.2]). Let $f : X \to (-\infty, \infty]$ be a proper, convex and lower semi-continuous function such that $\operatorname{argmin}_X f \neq \varnothing$, and $\{\lambda_n\}$ a sequence of positive real numbers. By Lemma 4.1, we have that $R_{\lambda_n f}$ is quasi-nonexpansive for all $n \in \mathbb{N}$. Let $\{x_n\}$ be a sequence in X and $p \in \bigcap_{n=1}^{\infty} \mathrm{Fix}(R_{\lambda_n f}) = \operatorname{argmin}_X f$ such that $\sup_{n \in \mathbb{N}} d(x_n, p) < \pi/2$ and

$$\lim_{n\to\infty} \frac{\cos d(x_n, p)}{\cos d(R_{\lambda_n f}x_n, p)} = 1.$$

It follows Lemma 4.1 that

$$\lim_{n\to\infty} \cos d(R_{\lambda_n f}x_n, x_n) \geq \lim_{n\to\infty} \frac{\cos d(x_n, p)}{\cos d(R_{\lambda_n f}x_n, p)} = 1.$$

Consequently, we obtain

$$\lim_{n\to\infty} d(R_{\lambda_n f}x_n, x_n) = 0.$$

Hence $\{R_{\lambda_n f}\}$ is a strongly quasi-nonexpansive sequence.

As a result of Theorem 3.1, we obtain a Δ-convergence theorem for the proximal point algorithm.

Theorem 4.4. *Let $f : X \to (-\infty, \infty]$ be a proper, convex and lower semi-continuous function such that $\operatorname{argmin}_X f \neq \varnothing$, and $\{\lambda_n\}$ a sequence of positive real numbers. Define a sequence $\{x_n\}$ in X by $x_1 \in X$ and*

$$x_{n+1} = R_{\lambda_n f} x_n \quad \text{for } n \in \mathbb{N}.$$

If $\liminf_n \lambda_n > 0$, then the sequence $\{x_n\}$ is Δ-convergent an element z of $\operatorname{argmin}_X f$. Moreover, $\{Px_n\}$ is strongly convergent to z, where P denotes the metric projection of X onto $\operatorname{argmin}_X f$.

As a consequence of Theorem 3.4, we also obtain a strong convergence theorem which improves Theorems 5.1 (ii) and 5.2 of [16] (see Theorem 1.1).

Theorem 4.5. *Let $f : X \to (-\infty, \infty]$ be a proper, convex and lower semi-continuous function such that $\operatorname{argmin} f \neq \varnothing$, and $\{\lambda_n\}$ a sequence of positive real numbers. Define a sequence $\{x_n\}$ in X by $x_1, u \in X$ and*

$$x_{n+1} = \alpha_n u \oplus (1 - \alpha_n) R_{\lambda_n f} x_n \quad \text{for } n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$ such that $\lim_n \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Suppose, in addition, that either $\operatorname{diam} X < \pi/2$ or $\sum_{n=1}^{\infty} \alpha_n^2 = \infty$. If $\liminf_n \lambda_n > 0$, then the sequence $\{x_n\}$ converges strongly to the point Pu, where P denotes the metric projection of X onto $\operatorname{argmin}_X f$.

4.2. **Application to the image recovery problem.** We know that the image recovery problem in a Hilbert space can be formulated as to find the nearest point in the intersection of a family of closed convex subsets. In this section, we generate an iterative sequence converging to the nearest point in the intersection of a countable family of closed convex subsets of a complete CAT(1) space from a given point by using the metric projection of each subsets.

In the following lemma, we show how to generate a countable family of the metric projections satisfying Condition (B).

Lemma 4.6. *Let $\{C_n\}$ be a countable family of closed convex subsets of X such that $\bigcap_{n=1}^{\infty} C_n \neq \varnothing$, and P_{C_n} be the metric projection from X onto C_n for all $n \in \mathbb{N}$. Define a sequence $\{S_n\}$ of mappings on X by*

$$\begin{aligned}
S_1 &:= P_{C_1};\\
S_2 &:= \frac{1}{2} P_{C_1} \oplus \frac{1}{2} P_{C_2};\\
S_3 &:= \frac{1}{2} P_{C_1} \oplus \frac{1}{2} \left(\frac{1}{2} P_{C_2} \oplus \frac{1}{2} P_{C_3} \right);\\
&\ \vdots\\
S_n &:= \frac{1}{2} P_{C_1} \oplus \frac{1}{2} \left(\frac{1}{2} P_{C_2} \oplus \frac{1}{2} \left(\cdots \oplus \frac{1}{2} \left(\frac{1}{2} P_{C_{n-1}} \oplus \frac{1}{2} P_{C_n} \right) \cdots \right) \right).
\end{aligned}$$

Then the sequence $\{S_n\}$ is a sequence of quasi-nonexpansive mappings and satisfies Condition (B).

To prove Lemma 4.6, we need the following lemma.

Lemma 4.7 ([18]). *Let S and T be quasi-nonexpansive mappings on X such that $\mathrm{Fix}(S)\cap\mathrm{Fix}(T)\neq\varnothing$. Then, for each $t\in(0,1)$, $\mathrm{Fix}(tS\oplus(1-t)T)=\mathrm{Fix}(S)\cap\mathrm{Fix}(T)$ and the mapping $tS\oplus(1-t)T$ is quasi-nonexpansive.*

Proof of Lemma 4.6. Note that the metric projection P_{C_n} is quasi-nonexpansive and Δ-demiclosed for all $n\in\mathbb{N}$. By Lemma 4.7, we have that S_n is quasi-nonexpansive for all $n\in\mathbb{N}$ such that $\bigcap_{n=1}^{\infty}\mathrm{Fix}(S_n)=\bigcap_{n=1}^{\infty}\mathrm{Fix}(P_{C_n})=\bigcap_{n=1}^{\infty}C_n\neq\varnothing$. Let $\{x_n\}$ is a sequence in X and $p\in\bigcap_{n=1}^{\infty}C_n$ such that $\limsup_n d(x_n,p)<\pi/2$ and

$$\lim_{n\to\infty} d(x_n,S_nx_n)=0.$$

Let z be the Δ-limit of $\{x_n\}$. For each $k\in\mathbb{N}$, put $U_k^{(k)}:=P_{C_k}$ and

$$U_n^k:=\frac{1}{2}P_{C_k}\oplus\frac{1}{2}\left(\frac{1}{2}P_{C_{k+1}}\oplus\frac{1}{2}\left(\cdots\oplus\frac{1}{2}\left(\frac{1}{2}P_{C_{n-1}}\oplus\frac{1}{2}P_{C_n}\right)\cdots\right)\right)$$

for all $n>k$. Note that U_n^k is quiasi-nonexpansive for all $k\in\mathbb{N}$ and $n\geq k$. Then we have the following estimates:

$$\begin{aligned}
&\cos d(S_nx_n,p)\sin d\left(P_{C_1}x_n,V_n^{(2)}x_n\right)\\
&=\cos d\left(\frac{1}{2}P_{C_1}x_n\oplus\frac{1}{2}U_n^{(2)}x_n,p\right)\sin d\left(P_{C_1}x_n,U_n^{(2)}x_n\right)\\
&\geq\left(\cos d(P_{C_1}x_n,p)+\cos d(U_n^{(2)}x_n,p)\right)\sin\frac{d\left(P_{C_1}x_n,U_n^{(2)}x_n\right)}{2}\\
&\geq(\cos d(x_n,p)+\cos d(x_n,p))\sin\frac{d\left(P_{C_1}x_n,U_n^{(2)}x_n\right)}{2}\\
&=2\cos d(x_n,p)\sin\frac{d\left(P_{C_1}x_n,U_n^{(2)}x_n\right)}{2}
\end{aligned}$$

for all $n\in\mathbb{N}$. This implies that

$$\cos\frac{d\left(P_{C_1}x_n,V_n^{(2)}x_n\right)}{2}\geq\frac{\cos d(x_n,p)}{\cos d(S_nx_n,p)}$$

for all $n\in\mathbb{N}$. Since $\lim_n d(x_n,S_nx_n)=0$ and $\limsup_n d(x_n,p)<\pi/2$,

$$\lim_{n\to\infty}\cos\frac{d\left(P_{C_1}x_n,V_n^{(2)}x_n\right)}{2}\geq\lim_{n\to\infty}\frac{\cos d(x_n,p)}{\cos d(S_nx_n,p)}=1.$$

Consequently, we obtain

$$\begin{aligned}
\lim_{n\to\infty} d(x_n,P_{C_1}x_n)&\leq\lim_{n\to\infty}(d(x_n,S_nx_n)+d(S_nx_n,P_{C_1}x_n))\\
&=\lim_{n\to\infty}\left(d(x_n,S_nx_n)+\frac{1}{2}d(V_n^{(2)}x_n,P_{C_1}x_n)\right)\\
&=0.
\end{aligned}$$

It follows from Condition (B) of $\{P_{C_1}\}$ that $x \in \mathrm{Fix}(P_{C_1})$. We also have

$$\begin{aligned}\lim_{n\to\infty} d(x_n, V_n^{(2)}x_n) &\le \lim_{n\to\infty}\left(d(x_n, S_nx_n) + d(S_nx_n, V_n^{(2)}x_n)\right)\\ &= \lim_{n\to\infty}\left(d(x_n, S_nx_n) + \frac{1}{2}d(P_{C_1}x_n, V_n^{(2)}x_n)\right)\\ &= 0.\end{aligned}$$

Continuing this procedure gives $x \in \bigcap_{n=1}^{\infty}\mathrm{Fix}(P_{C_n})$, that is, $\{S_n\}$ satisfies Condition (B). □

Using Lemma 4.6 together with Theorem 3.1, we obtain the following result.

Theorem 4.8. *Let $\{C_n\}$ be a countable family of closed convex subsets of X such that $\bigcap_{n=1}^{\infty} C_n \neq \varnothing$, and $\{S_n\}$ be as in Lemma 4.6. Define a sequence $\{x_n\}$ in X by $x_1 \in X$ and*

$$x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n)S_nx_n \quad \text{for } n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence in $[a,b] \subset (0,1)$. Then the sequence $\{x_n\}$ is Δ-convergent to an element z of $\bigcap_{n=1}^{\infty} C_n$. Moreover, $\{Px_n\}$ is strongly convergent to z, where P denotes the metric projection of X onto $\bigcap_{n=1}^{\infty} C_n$.

As a consequence of Theorem 3.4 and Lemma 4.6, we obtain a strong convergence theorem for image recovery problem for a countable family of closed convex subsets of the space.

Theorem 4.9. *Let $\{C_n\}$ be a countable family of closed convex subsets of X such that $\bigcap_{n=1}^{\infty} C_n \neq \varnothing$, and $\{S_n\}$ be as in Lemma 4.6. Define a sequence $\{x_n\}$ in X by $x_1, u \in X$ and*

$$x_{n+1} = \beta_n x_n \oplus (1-\beta_n)\left(\alpha_n u \oplus (1-\alpha_n)S_nx_n\right) \quad \text{for } n \in \mathbb{N},$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ such that $\lim_n \alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$, and $\{\beta_n\} \subset [a,b] \subset (0,1)$. If either $M < \pi/2$ or $\sum_{n=1}^{\infty}\alpha_n^2 = \infty$, then the sequence $\{x_n\}$ converges strongly to $P_{\bigcap_{n=1}^{\infty} C_n} u$.

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