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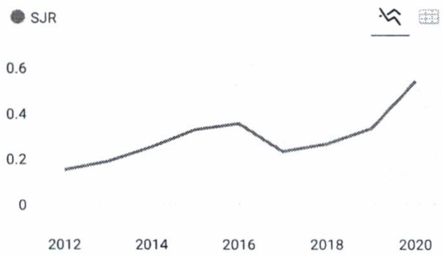

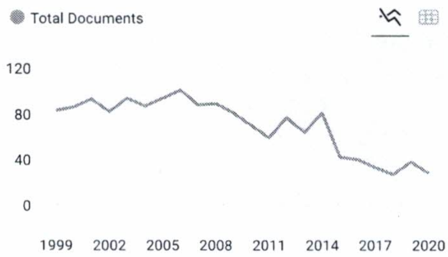

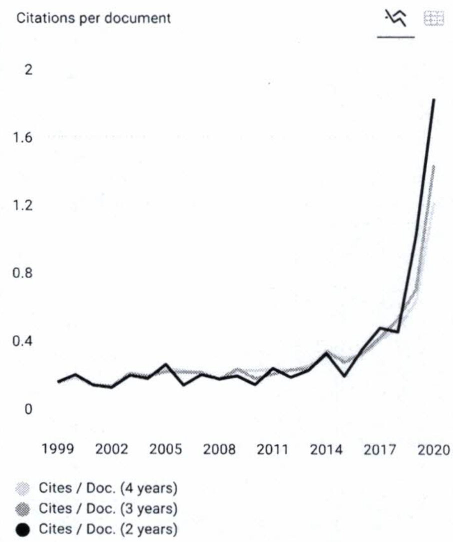

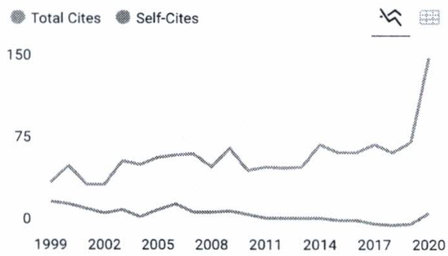

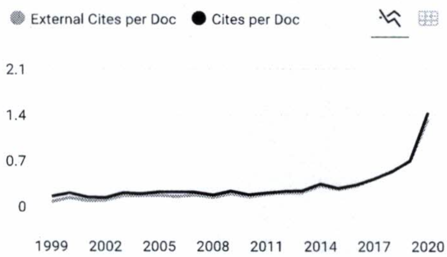

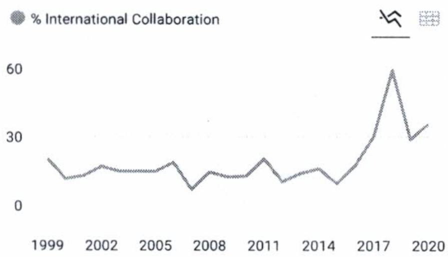

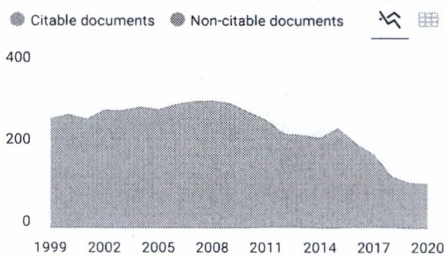

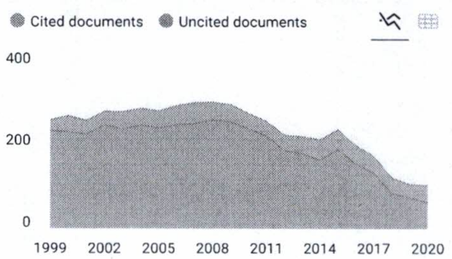

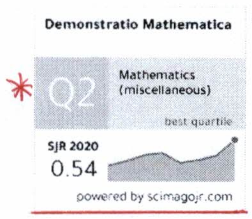

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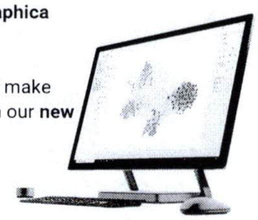

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Research Article

Nuttapol Pakkaranang*, Habib ur Rehman, and Wiyada Kumam

Two strongly convergent self-adaptive iterative schemes for solving pseudo-monotone equilibrium problems with applications

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Abstract: The aim of this paper is to propose two new modified extragradient methods to solve the pseudo-monotone equilibrium problem in a real Hilbert space with the Lipschitz-type condition. The iterative schemes use a new step size rule that is updated on each iteration based on the value of previous iterations. By using mild conditions on a bi-function, two strong convergence theorems are established. The applications of proposed results are studied to solve variational inequalities and fixed point problems in the setting of real Hilbert spaces. Many numerical experiments have been provided in order to show the algorithmic performance of the proposed methods and compare them with the existing ones.

Keywords: equilibrium problem, pseudomonotone bifunction, Lipschitz-type conditions, strong convergence, variational inequality problems, fixed point problem

MSC 2020: 47J25, 47H09, 47H06, 47J05

1 Introduction

Assume that $\mathcal{K}$ is a convex subset of a real Hilbert space $\mathcal{E}$. Suppose that $f : \mathcal{E} \times \mathcal{E} \to \mathbb{R}$ satisfying $f(y_1, y_1) = 0$ for each $y_1 \in \mathcal{K}$ and the *equilibrium problem* (EP) [1,2] for f on $\mathcal{K}$ is defined in the following manner:

$$\text{Find } x^* \in \mathcal{K} \text{ such that } f(x^*, y_1) \geq 0, \quad \text{for all } y_1 \in \mathcal{K}. \tag{EP}$$

Let the solution of an EP be denoted by $EP(f, \mathcal{K})$ and $x^* \in EP(f, \mathcal{K})$. Let take $x^* = P_{EP(f,\mathcal{K})}(\theta)$, where θ stands for the zero element in $\mathcal{E}$. Next, we consider the different types of bi-function monotonicity (see [1,3] for further information). A bi-function $f : \mathcal{E} \times \mathcal{E} \to \mathbb{R}$ on $\mathcal{K}$ for some $\xi > 0$ is said to be

(1) *strongly monotone* if

$$f(y_1, y_2) + f(y_2, y_1) \leq -\xi\|y_1 - y_2\|^2, \quad \forall y_1, y_2 \in \mathcal{K};$$

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(2) *monotone* if

$$f(y_1, y_2) + f(y_2, y_1) \leq 0, \quad \forall y_1, y_2 \in \mathcal{K};$$

(3) *strongly pseudo-monotone* if

$$f(y_1, y_2) \geq 0 \Rightarrow f(y_2, y_1) \leq -\xi\|y_1 - y_2\|^2, \quad \forall y_1, y_2 \in \mathcal{K};$$

(4) *pseudo-monotone* if

$$f(y_1, y_2) \geq 0 \Rightarrow f(y_2, y_1) \leq 0, \quad \forall y_1, y_2 \in \mathcal{K}.$$

Let $f : \mathcal{E} \times \mathcal{E} \to \mathbb{R}$ be satisfying the *Lipschitz-type condition* [4] on $\mathcal{K}$ if there exist two constants $c_1, c_2 > 0$, such that

$$f(y_1, y_3) \leq f(y_1, y_2) + f(y_2, y_3) + c_1\|y_1 - y_2\|^2 + c_2\|y_2 - y_3\|^2, \quad \forall y_1, y_2, y_3 \in \mathcal{K}.$$

The unique format of (EP) unifies a number of mathematical problems such as fixed-point problem, variational inequality problems, vector and scalar minimization problems, the complementarity problems, the saddle points problems, the Nash EP in non-cooperative games and the inverse optimization problems [1,5,6]. The problem (EP) is also taken as the Ky Fan inequality due to previous contributions [2]. Many iterative methods have been proposed and studied to solve different classes of (EP). Many effective iterative methods have been already established along with their convergence analysis [7–13] and others in [14–23].

The *regularization method* is the most effective technique to solve many ill-possessed problems in different fields of pure and applied mathematics. The key advantage of the regularization method is that it can solve monotone EPs and transform the original problem into a strongly monotone equilibrium sub-problem. Therefore, each sub-problem is strongly monotone and there is a unique solution. In particular, the sub-problem can be resolved more easily than the original problem, and the regularization solutions converge to some solution of the initial problem once the regularization variables appear to have an appropriate limit. The two common regularization methods are *proximal point method* and *Tikhonov's regularized method*. These two methods have recently been used to solve the (EP) [24–27].

The *proximal method* [28] is an effective method to solve EPs and need to solve minimization problems on each iterative step. This method was also known as the two-step extragradient method in [29] due to the previous contribution of Korpelevich extragradient method [30] to solve the saddle point problems. Tran et al. in [29] introduced a sequence of $\{x_n\}$ as follows:

$$\begin{cases} x_0 \in \mathcal{K}, \\ y_n = \arg\min\limits_{y \in \mathcal{K}} \left\{\zeta f(x_n, y) + \dfrac{1}{2}\|x_n - y\|^2\right\}, \\ x_{n+1} = \arg\min\limits_{y \in \mathcal{K}} \left\{\zeta f(y_n, y) + \dfrac{1}{2}\|x_n - y\|^2\right\}, \end{cases}$$

where $0 < \zeta < \min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\}$.

Recently, Wang et al. in [31] introduced a non-convex combination iterative method to solve pseudo-monotone EPs. Strong convergence of iterative sequences is the main contribution of the proposed method. The detailed method is as follows: choose $0 < \zeta_n < \min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\}$, $\delta_n \subset [\delta, 1)$ with $0 < \delta < 1$ and α_n such that

$$\lim_{n\to\infty} \alpha_n = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \alpha_n = \infty.$$

Compute $x_{n+1} = P_{\mathcal{K}}[\alpha_n x_n + (1 - \alpha_n)z_n - \alpha_n\delta_n x_n]$, where

$$\begin{cases} y_n = \arg\min\limits_{y \in \mathcal{K}} \left\{\zeta_n f(x_n, y) + \dfrac{1}{2}\|x_n - y\|^2\right\}, \\ z_n = \arg\min\limits_{y \in \mathcal{K}} \left\{\zeta_n f(y_n, y) + \dfrac{1}{2}\|x_n - y\|^2\right\}. \end{cases} \tag{1.1}$$

A natural question arises.

> Is it possible to introduce a modified version of method (1.1) in the sense that constant step size took over with the non-monotonic step size rule and it improve the numerical efficiency of the method"?

In this paper, we provide a positive answer to the above question, that is, the gradient method provides a strong convergence sequence by using a non-monotonic step size rule for solving EPs accompanied with pseudo-monotone bi-functions. Motivated by the works of Censor et al. [32] and Wang et al. [31] we introduce a new gradient-type method to figure out the problem (EP) in the setting of an infinite-dimensional real Hilbert space. Some applications on the topic of the variational inequality problem and the fixed point problem are provided. Numerical experiments have described that the proposed methods are more successful than the given one in [31].

The rest of this article has been arranged as follows: Section 2 contains some basic definitions and identities used in this paper. Sections 3 and 4 include the proposed methods as well as the convergence theorems. Section 5 presents an application of our results to solve variational inequalities and the fixed point problems. Section 6 sets out numerical explanations which demonstrate the computational effectiveness of the proposed methods.

2 Preliminaries

For a convex function $h : \mathcal{K} \to \mathbb{R}$ subdifferential of h at $y_1 \in \mathcal{K}$ is defined by

$$\partial h(y_1) = \{y_3 \in \mathcal{E} : h(y_2) - h(y_1) \geq \langle y_3, y_2 - y_1 \rangle, \quad \forall y_2 \in \mathcal{K}\}.$$

A *normal cone of* $\mathcal{K}$ at $y_1 \in \mathcal{K}$ is defined by

$$N_{\mathcal{K}}(y_1) = \{y_3 \in \mathcal{E} : \langle y_3, y_2 - y_1 \rangle \leq 0, \quad \forall y_2 \in \mathcal{K}\}.$$

Lemma 2.1. [33] *Assume that $h : \mathcal{K} \to \mathbb{R}$ is lower semi-continuous, convex and subdifferentiable function on $\mathcal{K}$. Then, $y_1 \in \mathcal{K}$ is a minimizer of a function h if and only if*

$$0 \in \partial h(y_1) + N_{\mathcal{K}}(y_1),$$

where $\partial h(y_1)$ and $N_{\mathcal{K}}(y_1)$ denote the subdifferential of h at $y_1 \in \mathcal{K}$ and the normal cone of $\mathcal{K}$ at y_1, respectively.

Definition 2.2. [34] The *metric projection* $P_{\mathcal{K}}(y_1)$ for $y_1 \in \mathcal{E}$ onto a closed and convex subset $\mathcal{K}$ of $\mathcal{E}$ is defined by

$$P_{\mathcal{K}}(y_1) = \arg\min\{\|y_2 - y_1\| : y_2 \in \mathcal{K}\}.$$

Lemma 2.3. [35] *Let $P_{\mathcal{K}} : \mathcal{E} \to \mathcal{K}$ be a metric projection on $\mathcal{K}$. Then*
(1) *For each $y_2 \in \mathcal{K}$ and $y_1 \in \mathcal{E}$ such that*

$$\|y_1 - P_{\mathcal{K}}(y_1)\| \leq \|y_1 - y_2\|^2;$$

(2) *$y_3 = P_{\mathcal{K}}(y_1)$ if and only if*

$$\langle y_1 - y_3, y_2 - y_3 \rangle \leq 0, \quad \forall y_2 \in \mathcal{K}.$$

Lemma 2.4. [36] *For each $y_1, y_2 \in \mathcal{E}$ with $\chi \in \mathbb{R}$, we have*

$$\|\chi y_1 + (1-\chi)y_2\|^2 = \chi\|y_1\|^2 + (1-\chi)\|y_2\|^2 - \chi(1-\chi)\|y_1 - y_2\|^2$$

and

$$\|y_1 + y_2\|^2 \leq \|y_1\|^2 + 2\langle y_2, y_1 + y_2 \rangle.$$

Lemma 2.5. [37] *Let a sequence $\{\chi_n\}$ of non-negative real numbers such that*

$$\chi_{n+1} \leq (1 - \tau_n)\chi_n + \tau_n\delta_n, \quad \forall n \in \mathbb{N},$$

where $\{\tau_n\} \subset (0, 1)$ and $\{\delta_n\} \subset \mathbb{R}$ satisfy the following conditions:

$$\lim_{n\to\infty} \tau_n = 0, \quad \sum_{n=1}^{\infty} \tau_n = \infty, \quad \text{and} \quad \limsup_{n\to\infty} \delta_n \leq 0.$$

Then, $\lim_{n\to\infty} \chi_n = 0$.

Lemma 2.6. [38] *Let $\{\chi_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $\chi_{n_i} < \chi_{n_{i+1}}$ for all $i \in \mathbb{N}$. Then, there exists a non-decreasing sequence $m_k \subset \mathbb{N}$ such that $m_k \to \infty$ as $k \to \infty$, and the following conditions are fulfilled by all (sufficiently large) numbers $k \in \mathbb{N}$:*

$$\chi_{m_k} \leq \chi_{m_{k+1}} \quad \text{and} \quad \chi_k \leq \chi_{m_{k+1}}.$$

In fact, $m_k = \max\{j \leq k : \chi_j \leq \chi_{j+1}\}$.

Let $f : \mathcal{E} \times \mathcal{E} \to \mathbb{R}$ satisfy the following conditions:
(C1) $f(y_2, y_2) = 0$ for all $y_2 \in \mathcal{K}$ and f is pseudo-monotone on $\mathcal{K}$;
(C2) f satisfies the Lipschitz-type conditions on $\mathcal{E}$ with constants $c_1 > 0$ and $c_2 > 0$;
(C3) $f(y_1, y_2)$ is jointly weakly continuous on $\mathcal{E} \times \mathcal{E}$;
(C4) $f(y_1, .)$ is subdifferentiable and convex over $\mathcal{E}$ for each $x \in \mathcal{E}$.

3 Explicit subgradient extragradient method and its convergence analysis

The following is the first method in detail.

Algorithm 1

Initialization: Let $x_1 \in \mathcal{K}$, $\sigma < \min\left\{1, \frac{1}{2c_1}, \frac{1}{2c_2}\right\}$, $\mu \in (0, \sigma)$, $\zeta_1 > 0$, $\delta_n \subset [\delta, 1)$ with $0 < \delta < 1$ and $\alpha_n \subset (0, 1)$ such that

$$\lim_{n\to\infty} \alpha_n = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \alpha_n = \infty.$$

Step 1: Compute

$$y_n = \arg\min_{y\in\mathcal{K}} \left\{\zeta_n f(x_n, y) + \frac{1}{2}\|x_n - y\|^2\right\}.$$

If $x_n = y_n$, then STOP. Otherwise, go to the next step.

Step 2: Construct a half-space

$$\mathcal{E}_n = \{z \in \mathcal{E} : \langle x_n - \zeta_n\omega_n - y_n, z - y_n\rangle \leq 0\},$$

where $\omega_n \in \partial f(x_n, y_n)$ and compute

$$z_n = \arg\min_{y\in\mathcal{E}_n} \left\{\mu\zeta_n f(y_n, y) + \frac{1}{2}\|x_n - y\|^2\right\}.$$

Step 3: Compute

$$x_{n+1} = P_{\mathcal{K}}[\alpha_n x_n + (1-\alpha_n)z_n - \alpha_n \delta_n x_n].$$

Step 4: Revise the step size rule as follows:

$$\zeta_{n+1} = \begin{cases} \min\left\{\sigma, \dfrac{\mu f(y_n, z_n)}{f(x_n, z_n) - f(x_n, y_n) - c_1\|x_n - y_n\|^2 - c_2\|z_n - y_n\|^2 + 1}\right\}, \\ \quad \text{if } \dfrac{\mu f(y_n, z_n)}{f(x_n, z_n) - f(x_n, y_n) - c_1\|x_n - y_n\|^2 - c_2\|z_n - y_n\|^2 + 1} > 0, \\ \sigma \quad \text{else.} \end{cases} \tag{3.1}$$

Put $n := n+1$ and go back to **Step 1**.

Remark 3.1. The iteration ζ_{n+1} in (3.1) is well-defined and

$$\zeta_{n+1}(f(x_n, z_n) - f(x_n, y_n) - c_1\|x_n - y_n\|^2 - c_2\|y_n - z_n\|^2) \le \mu f(y_n, z_n). \tag{3.2}$$

Lemma 3.2. *Let $f : \mathcal{E} \times \mathcal{E} \to \mathbb{R}$ be a bi-function satisfying conditions* (C1)–(C4). *Then, for any $x^* \in EP(f, \mathcal{K}) \ne \emptyset$, we have*

$$\|z_n - x^*\|^2 \le \|x_n - x^*\|^2 - (1-\zeta_{n+1})\|z_n - x_n\|^2 - \zeta_{n+1}(1 - 2c_1\zeta_n)\|x_n - y_n\|^2 - \zeta_{n+1}(1 - 2c_2\zeta_n)\|z_n - y_n\|^2.$$

Proof. The value of z_n and Lemma 2.1 give that

$$0 \in \partial_2\left\{\mu\zeta_n f(y_n, y) + \frac{1}{2}\|x_n - y\|^2\right\}(z_n) + N_{\mathcal{E}_n}(z_n).$$

From above $\omega_n \in \partial_2 f(y_n, z_n)$ and $\overline{\omega}_n \in N_{\mathcal{E}_n}(z_n)$, we have

$$\mu\zeta_n\omega_n + z_n - x_n + \overline{\omega}_n = 0.$$

Thus, we have since $\overline{\omega}_n \in N_{\mathcal{E}_n}(z_n)$, $\langle \overline{\omega}_n, y - z_n\rangle \le 0$, $\forall y \in \mathcal{E}_n$. It implies that

$$\mu\zeta_n\langle \omega_n, y - z_n\rangle \ge \langle x_n - z_n, y - z_n\rangle, \quad \forall y \in \mathcal{E}_n. \tag{3.3}$$

By $\omega_n \in \partial_2 f(y_n, z_n)$, we obtain

$$f(y_n, y) - f(y_n, z_n) \ge \langle \omega_n, y - z_n\rangle, \quad \forall y \in \mathcal{E}. \tag{3.4}$$

Combining expressions (3.3) and (3.4) gives that

$$\mu\zeta_n f(y_n, y) - \mu\zeta_n f(y_n, z_n) \ge \langle x_n - z_n, y - z_n\rangle, \quad \forall y \in \mathcal{E}_n. \tag{3.5}$$

Substitution by $y = x^*$ in (3.5), we get

$$\mu\zeta_n f(y_n, x^*) - \mu\zeta_n f(y_n, z_n) \ge \langle x_n - z_n, x^* - z_n\rangle, \quad \forall y \in \mathcal{E}_n. \tag{3.6}$$

Since $x^* \in EP(f, \mathcal{K})$, $f(x^*, y_n) \ge 0$ and because of condition (C1) we have $f(y_n, x^*) \le 0$. It implies that

$$\langle x_n - z_n, z_n - x^*\rangle \ge \mu\zeta_n f(y_n, z_n). \tag{3.7}$$

Combining expressions (3.2) and (3.7), we have

$$\langle x_n - z_n, z_n - x^*\rangle \ge \zeta_{n+1}[\zeta_n\{f(x_n, z_n) - f(x_n, y_n)\} - c_1\zeta_n\|x_n - y_n\|^2 - c_2\zeta_n\|z_n - y_n\|^2]. \tag{3.8}$$

Due to $z_n \in \mathcal{E}_n$, we have

$$\zeta_n\langle \omega_n, z_n - y_n\rangle \geq \langle x_n - y_n, z_n - y_n\rangle. \tag{3.9}$$

From $\omega_n \in \partial_2 f(x_n, y_n)$ and $y = z_n$, we have

$$f(x_n, z_n) - f(x_n, y_n) \geq \langle \omega_n, z_n - y_n\rangle, \quad \forall y \in \mathcal{E}. \tag{3.10}$$

Combining expressions (3.9) and (3.10), we obtain

$$\zeta_n\{f(x_n, z_n) - f(x_n, y_n)\} \geq \langle x_n - y_n, z_n - y_n\rangle. \tag{3.11}$$

Expressions (3.8) and (3.11) imply that

$$2\langle x_n - z_n, z_n - x^*\rangle \geq \zeta_{n+1}[2\langle x_n - y_n, z_n - y_n\rangle - 2c_1\zeta_n\|x_n - y_n\|^2 - 2c_2\zeta_n\|z_n - y_n\|^2]. \tag{3.12}$$

We have the following facts:

$$2\langle x_n - z_n, z_n - x^*\rangle = \|x_n - x^*\|^2 - \|z_n - x_n\|^2 - \|z_n - x^*\|^2.$$
$$2\langle x_n - y_n, z_n - y_n\rangle = \|x_n - y_n\|^2 + \|z_n - y_n\|^2 - \|x_n - z_n\|^2.$$

Combining above inequalities with (3.12) completes the proof. □

Theorem 3.3. *Let $\{x_n\}$ be a sequence generated by* Algorithm 1 *and the solution set $EP(f, \mathcal{K})$ is non-empty. Then, the sequence $\{x_n\}$ is strongly convergent to an element $x^* \in EP(f, \mathcal{K})$.*

Proof. From Lemma 3.2, we have

$$\|z_n - x^*\|^2 \leq \|x_n - x^*\|^2, \quad \forall n \geq 2. \tag{3.13}$$

Next, we prove that $\{x_n\}$ is bounded. For all $n \geq 2$, with Lemma 2.4 (i), we obtain

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|P_{\mathcal{K}}[\alpha_n(1-\delta_n)x_n + (1-\alpha_n)z_n] - P_{\mathcal{K}}(x^*)\|^2\\
&\leq \|\alpha_n(1-\delta_n)x_n + (1-\alpha_n)z_n - x^*\|^2\\
&= \|\alpha_n[(1-\delta_n)x_n - x^*] + (1-\alpha_n)(z_n - x^*)\|^2\\
&\leq \alpha_n\|(1-\delta_n)x_n - x^*\|^2 + (1-\alpha_n)\|z_n - x^*\|^2\\
&\leq \alpha_n[\|(1-\delta_n)(x_n - x^*) + \delta_n x^*\|^2] + (1-\alpha_n)\|x_n - x^*\|^2\\
&\quad - (1-\alpha_n)[(1-\zeta_{n+1})\|z_n - x_n\|^2 + \zeta_{n+1}(1-2c_1\zeta_n)\|x_n - y_n\|^2 + \zeta_{n+1}(1-2c_2\zeta_n)\|z_n - y_n\|^2]\\
&\leq \alpha_n[(1-\delta_n)\|x_n - x^*\|^2 + \delta_n\|x^*\|^2] + (1-\alpha_n)\|x_n - x^*\|^2\\
&\quad - (1-\alpha_n)[(1-\sigma)\|z_n - x_n\|^2 + \sigma(1-2c_1\sigma)\|x_n - y_n\|^2 + \sigma(1-2c_2\sigma)\|z_n - y_n\|^2]
\end{aligned} \tag{3.14}$$

$$\begin{aligned}
&= (1-\alpha_n\delta_n)\|x_n - x^*\|^2 + \alpha_n\delta_n\|x^*\|^2\\
&\leq \max\{\|x_n - x^*\|^2, \|x^*\|^2\} \leq \max\{\|x_2 - x^*\|^2, \|x^*\|^2\}.
\end{aligned} \tag{3.15}$$

Thus, $\{x_n\}$ is a bounded sequence as well as $\{y_n\}$, $\{z_n\}$ bounded. Let $q_n = \alpha_n x_n + (1-\alpha_n)z_n$, for every $n \in \mathbb{N}$. By Lemma 2.4(i), we have

$$\|q_n - x^*\|^2 = \|\alpha_n x_n + (1-\alpha_n)z_n - x^*\|^2 \leq \|x_n - x^*\|^2, \quad \forall n \geq 2. \tag{3.16}$$

Thus, we obtain

$$x_{n+1} = P_{\mathcal{K}}(q_n - \alpha_n\delta_n x_n) = P_{\mathcal{K}}[(1-\alpha_n\delta_n)q_n + \alpha_n\delta_n(1-\alpha_n)(z_n - x_n)]. \tag{3.17}$$

By Lemma 2.4(ii) and (3.16), (3.17), we have (see equation (3.6) [31])

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|P_{\mathcal{K}}[(1-\alpha_n\delta_n)q_n + \alpha_n\delta_n(1-\alpha_n)(z_n - x_n)] - P_{\mathcal{K}}(x^*)\|^2\\
&\leq (1-\alpha_n\delta_n)\|x_n - x^*\|^2 + 2\alpha_n\delta_n(1-\alpha_n)\langle z_n - x_n, (1-\alpha_n\delta_n)q_n + \alpha_n\delta_n(1-\alpha_n)(z_n - x_n) - x^*\rangle\\
&\quad + 2\alpha_n\delta_n(1-\alpha_n)\langle -x^*, z_n - x_n\rangle + 2\alpha_n\delta_n\langle -x^*, x_n - x^*\rangle + 2\alpha_n^2\delta_n^2\langle x^*, x_n\rangle.
\end{aligned} \tag{3.18}$$

The rest of the proof will be divided into the following two parts:

Case 1. Assume there is $n_2 \in \mathbb{N}$ ($n_2 \geq 2$) such that

$$\|x_{n+1} - x^*\| \leq \|x_n - x^*\|, \quad \forall n \geq n_2.$$

This implies that $\lim_{n\to\infty}\|x_n - x^*\|$ exists and let $\lim_{n\to\infty}\|x_n - x^*\| = l$. Thus, expression (3.14) ($\forall n \geq n_2$) implies that

$$\begin{aligned}&(1-\sigma)\|z_n - x_n\|^2 + \sigma(1-2c_1\sigma)\|x_n - y_n\|^2 + \sigma(1-2c_2\sigma)\|z_n - y_n\|^2\\&\leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \alpha_n\|x^*\|^2 + \alpha_n A_0,\end{aligned} \tag{3.19}$$

where A_0 is the finite number

$$A_0 = \sup\{[(1-\sigma)\|z_n - x_n\|^2 + \sigma(1-2c_1\sigma)\|x_n - y_n\|^2 + \sigma(1-2c_2\sigma)\|z_n - y_n\|^2] : \forall n \in \mathbb{N}\}.$$

The existence of $\lim_{n\to\infty}\|x_n - x^*\| = l$, with expression (3.19) implies that

$$\lim_{n\to\infty}\|x_n - z_n\| = \lim_{n\to\infty}\|x_n - y_n\| = \lim_{n\to\infty}\|z_n - y_n\| = 0. \tag{3.20}$$

Since the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to some $x \in \mathcal{K}$ and

$$\limsup_{n\to\infty}\langle -x^*, x_n - x^*\rangle = \limsup_{k\to\infty}\langle -x^*, x_{n_k} - x^*\rangle = \langle -x^*, x - x^*\rangle. \tag{3.21}$$

By expression (3.20), the subsequences $\{y_{n_k}\}$ and $\{z_{n_k}\}$ weakly converge to x as $k \to \infty$. From (3.5), we have

$$\mu\zeta_{n_k} f(y_{n_k}, y) - \mu\zeta_{n_k} f(y_{n_k}, z_{n_k}) \geq \langle x_{n_k} - z_{n_k}, y - z_{n_k}\rangle, \quad \forall y \in \mathcal{E}_n. \tag{3.22}$$

By letting $k \to \infty$, it implies that

$$f(x, y) \geq 0, \quad \forall y \in \mathcal{K} \subset \mathcal{E}_n. \tag{3.23}$$

It follows that $x \in EP(f, \mathcal{K})$. In conclusion, by (3.21) and Lemma 2.3(ii), we get

$$\begin{aligned}\limsup_{n\to\infty}\langle -x^*, x_n - x^*\rangle &= \limsup_{k\to\infty}\langle -x^*, x_{n_k} - x^*\rangle = \langle -x^*, x - x^*\rangle\\&= \langle \theta - P_{EP(f,\mathcal{K})}(\theta), x - P_{EP(f,\mathcal{K})}(\theta)\rangle \leq 0.\end{aligned} \tag{3.24}$$

By using expressions (3.18), (3.20), (3.24) and Lemma 2.5, we obtain the required result.

Case 2. Suppose that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\|x_{n_i} - x^*\| \leq \|x_{n_{i+1}} - x^*\|, \quad \forall i \in \mathbb{N}.$$

From Lemma 2.6, there is a sequence $\{m_k\} \subset \mathbb{N}$ with $\{m_k\} \to \infty$, such that

$$\|x_{m_k} - x^*\| \leq \|x_{m_{k+1}} - x^*\| \quad \text{and} \quad \|x_k - x^*\| \leq \|x_{m_{k+1}} - x^*\|, \quad \text{for all } k \in \mathbb{N}. \tag{3.25}$$

From expression (3.19) (for all $m_k \geq 2$), we have

$$\begin{aligned}&(1-\sigma)\|z_{m_k} - x_{m_k}\|^2 + \sigma(1-2c_1\sigma)\|x_{m_k} - y_{m_k}\|^2 + \sigma(1-2c_2\sigma)\|z_{m_k} - y_{m_k}\|^2\\&\leq \|x_{m_k} - x^*\|^2 - \|x_{m_k+1} - x^*\|^2 + \alpha_{m_k}\|x^*\|^2 + \alpha_{m_k}A_0.\end{aligned} \tag{3.26}$$

The aforementioned expression implies that

$$\lim_{k\to\infty}\|z_{m_k} - x_{m_k}\| = \lim_{k\to\infty}\|x_{m_k} - y_{m_k}\| = \lim_{k\to\infty}\|z_{m_k} - y_{m_k}\| = 0. \tag{3.27}$$

Similar to expression (3.24), we have

$$\limsup_{k\to\infty}\langle -x^*, x_{m_k} - x^*\rangle \leq 0. \tag{3.28}$$

From expression (3.18), we obtain

$$\begin{aligned}\|x_{m_k+1}-x^*\|^2 \le (1-\alpha_{m_k}\delta_{m_k})\|x_{m_k}-x^*\|^2 &+ 2\alpha_{m_k}\delta_{m_k}(1-\alpha_{m_k})\langle z_{m_k}-x_{m_k}, (1-\alpha_{m_k}\delta_{m_k})q_{m_k}\\ &+ \alpha_{m_k}\delta_{m_k}(1-\alpha_{m_k})(z_{m_k}-x_{m_k})-x^*\rangle + 2\alpha_{m_k}\delta_{m_k}(1-\alpha_{m_k})\langle -x^*, z_{m_k}-x_{m_k}\rangle\\ &+ 2\alpha_{m_k}\delta_{m_k}\langle -x^*, x_{m_k}-x^*\rangle + 2\alpha_{m_k}^2\delta_{m_k}^2\langle x^*, x_{m_k}\rangle.\end{aligned} \tag{3.29}$$

It is given that $\|x_{m_k}-x^*\| \le \|x_{m_{k+1}}-x^*\|$. Thus, we have

$$\begin{aligned}\|x_{m_k+1}-x^*\|^2 \le (1-\alpha_{m_k}\delta_{m_k})\|x_{m_{k+1}}-x^*\|^2 &+ 2\alpha_{m_k}\delta_{m_k}(1-\alpha_{m_k})\langle z_{m_k}-x_{m_k}, (1-\alpha_{m_k}\delta_{m_k})q_{m_k}\\ &+ \alpha_{m_k}\delta_{m_k}(1-\alpha_{m_k})(z_{m_k}-x_{m_k})-x^*\rangle + 2\alpha_{m_k}\delta_{m_k}(1-\alpha_{m_k})\langle -x^*, z_{m_k}-x_{m_k}\rangle\\ &+ 2\alpha_{m_k}\delta_{m_k}\langle -x^*, x_{m_k}-x^*\rangle + 2\alpha_{m_k}^2\delta_{m_k}^2\langle x^*, x_{m_k}\rangle.\end{aligned} \tag{3.30}$$

Expressions (3.25) and (3.30) imply that

$$\begin{aligned}\|x_k-x^*\|^2 &\le \|x_{m_k+1}-x^*\|^2\\ &\le 2(1-\alpha_{m_k})\langle z_{m_k}-x_{m_k}, (1-\alpha_{m_k}\delta_{m_k})q_{m_k} + \alpha_{m_k}\delta_{m_k}(1-\alpha_{m_k})(z_{m_k}-x_{m_k})-x^*\rangle\\ &\quad + 2(1-\alpha_{m_k})\langle -x^*, z_{m_k}-x_{m_k}\rangle + 2\langle -x^*, x_{m_k}-x^*\rangle + 2\alpha_{m_k}\delta_{m_k}\langle x^*, x_{m_k}\rangle, \quad \forall n \ge 2.\end{aligned} \tag{3.31}$$

Since $\alpha_{m_k} \to 0$, it follows from (3.27) such that

$$\lim_{n\to\infty} \|x_k-x^*\|^2 \le \lim_{n\to\infty} \|x_{m_k+1}-x^*\|^2 \le 0. \tag{3.32}$$

Consequently, $x_n \to x^*$. This completes the proof. □

4 Modified explicit subgradient extragradient method and its convergence analysis

The following is the second method in detail.

Algorithm 2

Initialization: Let $x_1 \in \mathcal{E}$, $\sigma < \min\left\{1, \frac{1}{2c_1}, \frac{1}{2c_2}\right\}$, $\mu \in (0,\sigma)$, $\zeta_1 > 0$, $\delta_n \subset [\delta, 1)$ with $0 < \delta < 1$ and $\alpha_n, \beta_n \subset (0,1)$ such that

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty \quad \text{and} \quad \liminf_{n\to\infty} \beta_n(1-\beta_n) > 0.$$

Step 1: Compute

$$y_n = \arg\min_{y\in\mathcal{K}} \left\{\zeta_n f(P_{\mathcal{K}}(x_n), y) + \frac{1}{2}\|P_{\mathcal{K}}(x_n) - y\|^2\right\}.$$

If $x_n = y_n$, then STOP. Otherwise, go to the next step.

Step 2: Construct a half space first

$$\mathcal{E}_n = \{z \in \mathcal{E} : \langle P_{\mathcal{K}}(x_n) - \zeta_n\omega_n - y_n, z - y_n\rangle \le 0\},$$

where $\omega_n \in \partial f(P_{\mathcal{K}}(x_n), y_n)$ and solve the following convex problem:

$$z_n = \arg\min_{y\in\mathcal{E}_n} \left\{\mu\zeta_n f(y_n, y) + \frac{1}{2}\|P_{\mathcal{K}}(x_n) - y\|^2\right\}.$$

Step 3: Compute

$$x_{n+1} = \alpha_n(1-\delta_n)x_n + (1-\alpha_n)[\beta_n z_n + (1-\beta_n)x_n].$$

Step 4: Revising the step size as follows:

$$\zeta_{n+1} = \begin{cases} \min\left\{\sigma, \dfrac{\mu f(y_n, z_n)}{f(P_{\mathcal{K}}(x_n), z_n) - f(P_{\mathcal{K}}(x_n), y_n) - c_1\|P_{\mathcal{K}}(x_n) - y_n\|^2 - c_2\|z_n - y_n\|^2 + 1}\right\}, \\ \quad \text{if } \dfrac{\mu f(y_n, z_n)}{f(P_{\mathcal{K}}(x_n), z_n) - f(P_{\mathcal{K}}(x_n), y_n) - c_1\|P_{\mathcal{K}}(x_n) - y_n\|^2 - c_2\|z_n - y_n\|^2 + 1} > 0, \\ \sigma \qquad \text{else.} \end{cases} \tag{4.1}$$

Set $n := n + 1$ and go back to **Step 1**.

Remark 4.1. The iteration ζ_{n+1} in (4.1) is well-defined and

$$\zeta_{n+1}(f(P_{\mathcal{K}}(x_n), z_n) - f(P_{\mathcal{K}}(x_n), y_n) - c_1\|P_{\mathcal{K}}(x_n) - y_n\|^2 - c_2\|y_n - z_n\|^2) \le \mu f(y_n, z_n). \tag{4.2}$$

Lemma 4.2. *Let $f : \mathcal{E} \times \mathcal{E} \to \mathbb{R}$ be a bi-function satisfying conditions* (C1)–(C4). *Then, for each $x^* \in EP(f, \mathcal{K}) \neq \emptyset$, we have*

$$\|z_n - x^*\|^2 \le \|P_{\mathcal{K}}(x_n) - x^*\|^2 - (1 - \zeta_{n+1})\|z_n - P_{\mathcal{K}}(x_n)\|^2 - \zeta_{n+1}(1 - 2c_1\zeta_n)\|P_{\mathcal{K}}(x_n) - y_n\|^2 - \zeta_{n+1}(1 - 2c_2\zeta_n)\|z_n - y_n\|^2.$$

Theorem 4.3. *Let $\{x_n\}$ be a sequence generated by* Algorithm 2 *and the solution set $EP(f, \mathcal{K})$ is non-empty. Then, the sequence $\{x_n\}$ is strongly convergent to an element $x^* \in EP(f, \mathcal{K})$.*

Proof. First, we need to show that $\{x_n\}$ is a bounded sequence. By the use of Lemma 4.2, we have

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \|\alpha_n(1-\delta_n)x_n + (1-\alpha_n)[\beta_n z_n + (1-\beta_n)x_n] - x^*\|^2 \\ &= \|\alpha_n[(1-\delta_n)x_n - x^*] + (1-\alpha_n)[\beta_n(z_n - x^*) + (1-\beta_n)(x_n - x^*)]\|^2 \\ &\le \alpha_n\|(1-\delta_n)x_n - x^*\|^2 + (1-\alpha_n)\|\beta_n(z_n - x^*) + (1-\beta_n)(x_n - x^*)\|^2 \\ &\le \alpha_n\|(1-\delta_n)(x_n - x^*) + \delta_n x^*\|^2 + (1-\alpha_n)[\beta_n\|z_n - x^*\|^2 + (1-\beta_n)\|x_n - x^*\|^2 \\ &\quad - \beta_n(1-\beta_n)\|z_n - x_n\|^2] \\ &\le \alpha_n[(1-\delta_n)\|x_n - x^*\|^2 + \delta_n\|x^*\|^2] + (1-\alpha_n)[\|x_n - x^*\|^2 - \beta_n(1-\zeta_{n+1})\|z_n - P_{\mathcal{K}}(x_n)\|^2 \\ &\quad - \beta_n\zeta_{n+1}(1 - 2c_1\zeta_n)\|P_{\mathcal{K}}(x_n) - y_n\|^2 - \beta_n\zeta_{n+1}(1 - 2c_2\zeta_n)\|z_n - y_n\|^2 - \beta_n(1-\beta_n)\|z_n - x_n\|^2] \\ &\le (1 - \alpha_n\delta_n)\|x_n - x^*\|^2 + \alpha_n\delta_n\|x^*\|^2 - (1-\alpha_n)[\beta_n(1-\sigma)\|z_n - P_{\mathcal{K}}(x_n)\|^2 \\ &\quad + \beta_n\sigma(1 - 2c_1\sigma)\|P_{\mathcal{K}}(x_n) - y_n\|^2 + \beta_n\sigma(1 - 2c_2\sigma)\|z_n - y_n\|^2 + \beta_n(1-\beta_n)\|z_n - x_n\|^2]. \end{aligned} \tag{4.3}$$

Since $0 < \sigma < 1$, the above inequality implies that

$$\|x_{n+1} - x^*\|^2 \le \max\{\|x_n - x^*\|^2, \|x^*\|^2\} \le \max\{\|x_2 - x^*\|^2, \|x^*\|^2\}. \tag{4.4}$$

It implies that $\{x_n\}$ is a bounded sequence. By using x_{n+1} with Lemma 2.4 gives that (see equation (3.17) [31]):

$$\|x_{n+1} - x^*\|^2 \le (1 - \alpha_n\delta_n)\|x_n - x^*\|^2 + 2\alpha_n\delta_n(1-\alpha_n)\beta_n\langle z_n - x_n, x_{n+1} - x^*\rangle + 2\alpha_n\delta_n\langle -x^*, x_{n+1} - x^*\rangle. \tag{4.5}$$

Case 1. Assume that there is an $m_2 \in \mathbb{N}$ ($m_2 \ge 2$) such that

$$\|x_{n+1} - x^*\| \le \|x_n - x^*\|, \quad \forall n \ge m_2. \tag{4.6}$$

Then, the $\lim_{n\to\infty}\|x_n - x^*\|$ exists. By expression (4.3), we have

$$\begin{aligned}&\beta_n[(1-\sigma)\|z_n - P_{\mathcal{K}}(x_n)\|^2 + \sigma(1-2c_1\sigma)\|P_{\mathcal{K}}(x_n) - y_n\|^2 + \sigma(1-2c_2\sigma)\|z_n - y_n\|^2 + (1-\beta_n)\|z_n - x_n\|^2]\\ &\quad\le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \alpha_n\|x^*\|^2 + \alpha_n B_0,\end{aligned}\tag{4.7}$$

where B_0 is the finite number

$$\begin{aligned}B_0 = \sup\{&\beta_n[(1-\sigma)\|z_n - P_{\mathcal{K}}(x_n)\|^2 + \sigma(1-2c_1\sigma)\|P_{\mathcal{K}}(x_n) - y_n\|^2 + \sigma(1-2c_2\sigma)\|z_n - y_n\|^2\\ &+ (1-\beta_n)\|z_n - x_n\|^2] : \forall n \in \mathbb{N}\}.\end{aligned}\tag{4.8}$$

Thus, expression (4.7) implies that

$$\lim_{n\to\infty}\|P_{\mathcal{K}}(x_n) - z_n\| = \lim_{n\to\infty}\|P_{\mathcal{K}}(x_n) - y_n\| = \lim_{n\to\infty}\|z_n - y_n\| = 0 = \lim_{n\to\infty}\|x_n - z_n\| = 0.\tag{4.9}$$

Due to the boundedness of the sequence $\{x_n\}$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ weakly converges to $x \in \mathcal{K}$ and

$$\limsup_{n\to\infty}\langle -x^*, x_n - x^*\rangle = \limsup_{k\to\infty}\langle -x^*, x_{n_k} - x^*\rangle = \langle -x^*, x - x^*\rangle.\tag{4.10}$$

Similar to expression (3.5), we have

$$\mu\zeta_{n_k} f(y_{n_k}, y) - \mu\zeta_{n_k} f(y_{n_k}, z_{n_k}) \ge \langle P_{\mathcal{K}}(x_{n_k}) - z_{n_k}, y - z_{n_k}\rangle, \quad \forall y \in \mathcal{E}_n.\tag{4.11}$$

By letting $k \to +\infty$, we have

$$f(x, y) \ge 0, \quad \forall y \in \mathcal{K}.$$

It implies that $x \in EP(f, \mathcal{K})$. By expression (4.10) and Lemma 2.3, we obtain

$$\begin{aligned}\limsup_{n\to\infty}\langle -x^*, x_n - x^*\rangle &= \limsup_{k\to\infty}\langle -x^*, x_{n_k} - x^*\rangle = \langle -x^*, x - x^*\rangle\\ &= \langle \theta - P_{EP(f,\mathcal{K})}(\theta), x - P_{EP(f,\mathcal{K})}(\theta)\rangle \le 0.\end{aligned}\tag{4.12}$$

By expression (4.5) and Lemma 2.5, we obtain the required result.

Case 2. Suppose that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\|x_{n_i} - x^*\| \le \|x_{n_{i+1}} - x^*\|, \quad \forall i \in \mathbb{N}.$$

Then, by Lemma 2.6 there is a non-decreasing sequence $\{m_k\} \subset \mathbb{N}$ with $m_k \to \infty$, which implies that

$$\|x_{m_k} - x^*\| \le \|x_{m_{k+1}} - x^*\| \quad \text{and} \quad \|x_k - x^*\| \le \|x_{m_{k+1}} - x^*\|, \quad \text{for all} \quad k \in \mathbb{N}.\tag{4.13}$$

From expression (4.5), we obtain

$$\begin{aligned}\|x_{m_k+1} - x^*\|^2 \le{}& (1 - \alpha_{m_k}\delta_{m_k})\|x_{m_k} - x^*\|^2 + 2\alpha_{m_k}\delta_{m_k}(1-\alpha_{m_k})\beta_{m_k}\langle z_{m_k} - x_{m_k}, x_{m_k+1} - x^*\rangle\\ &+ 2\alpha_{m_k}\delta_{m_k}\langle -x^*, x_{m_k+1} - x^*\rangle.\end{aligned}\tag{4.14}$$

The remaining part is similar to Case 2 in Theorem 3.3. This completes the proof. □

5 Applications

Now, we study the applications of our main results to solve fixed point problems. An operator $T : \mathcal{K} \subset \mathcal{E} \to \mathcal{K}$ is said to be

(i) κ-strict pseudocontraction [39] on $\mathcal{K}$ if

$$\|Ty_1 - Ty_2\|^2 \le \|y_1 - y_2\|^2 + \kappa\|(y_1 - Ty_1) - (y_2 - Ty_2)\|^2, \quad \forall y_1, y_2 \in \mathcal{K},$$

which is equivalent to

$$\langle Ty_1 - Ty_2, y_1 - y_2\rangle \le \|y_1 - y_2\|^2 - \frac{1-\kappa}{2}\|(y_1 - Ty_1) - (y_2 - Ty_2)\|^2, \quad \forall y_1, y_2 \in \mathcal{K}.$$

(ii) Weakly sequentially continuous on $\mathcal{K}$ if

$$T(x_n) \rightharpoonup T(p) \text{ for any sequence in } \mathcal{K} \text{ satisfying } x_n \rightharpoonup p.$$

Corollary 5.1. *Assume that $\mathcal{K}$ is a non-empty, convex and closed subset of a Hilbert space $\mathcal{E}$ and $T : \mathcal{K} \to \mathcal{K}$ is a κ-strict pseudo-contraction and weakly continuous with* $\mathrm{Fix}(T) \neq \emptyset$. *Choose $x_1 \in \mathcal{K}$, $\sigma < \min\left\{1, \frac{1-\kappa}{3-2\kappa}\right\}$, $\mu \in (0, \sigma)$, $\zeta_1 > 0$, $\delta_n \subset [\delta, 1)$ with $0 < \delta < 1$ and $\alpha_n \subset (0, 1)$ such that*

$$\lim_{n\to\infty} \alpha_n = 0 \quad \textit{and} \quad \sum_{n=1}^{\infty} \alpha_n = \infty.$$

Compute $x_{n+1} = P_{\mathcal{K}}[\alpha_n x_n + (1-\alpha_n)z_n - \alpha_n\delta_n x_n]$, where

$$\begin{cases} y_n = (1-\zeta_n)x_n + \zeta_n T(x_n), \\ z_n = P_{\mathcal{E}_n}[x_n - \mu\zeta_n(y_n - T(y_n))], \end{cases}$$

where $\mathcal{E}_n = \{z \in \mathcal{E} : \langle (1-\zeta_n)x_n + \zeta_n T(x_n) - y_n, z - y_n\rangle \le 0\}$. Compute

$$\zeta_{n+1} = \begin{cases} \min\left\{\sigma, \dfrac{\mu\langle y_n - Ty_n, z_n - y_n\rangle}{\langle x_n - T(x_n), z_n - y_n\rangle - \left(\frac{3-2\kappa}{2-2\kappa}\right)\|x_n - y_n\|^2 - \left(\frac{3-2\kappa}{2-2\kappa}\right)\|z_n - y_n\|^2 + 1}\right\}, \\ \quad \textit{if} \quad \dfrac{\mu\langle y_n - Ty_n, z_n - y_n\rangle}{\langle x_n - T(x_n), z_n - y_n\rangle - \left(\frac{3-2\kappa}{2-2\kappa}\right)\|x_n - y_n\|^2 - \left(\frac{3-2\kappa}{2-2\kappa}\right)\|z_n - y_n\|^2 + 1} > 0, \\ \sigma \quad \textit{else}. \end{cases}$$

Then, the sequence $\{x_n\}$ converges strongly to an element $x^* = P_{Fix(T)}(\theta)$.

Corollary 5.2. *Assume that $\mathcal{K}$ is a non-empty, convex and closed subset of a Hilbert space $\mathcal{E}$ with $T : \mathcal{K} \to \mathcal{K}$ is a κ-strict pseudocontraction and weakly continuous with* $\mathrm{Fix}(T) \neq \emptyset$. *Choose $x_1 \in \mathcal{E}$, $\sigma < \min\left\{1, \frac{1-\kappa}{3-2\kappa}\right\}$, $\mu \in (0, \sigma)$, $\zeta_1 > 0$, $\delta_n \subset [\delta, 1)$ with $0 < \delta < 1$ and $\alpha_n, \beta_n \subset (0, 1)$ such that*

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty \quad \textit{and} \quad \liminf_{n\to\infty} \beta_n(1-\beta_n) > 0.$$

Compute $x_{n+1} = \alpha_n(1-\delta_n)x_n + (1-\alpha_n)[\beta_n z_n + (1-\beta_n)x_n]$, where

$$\begin{cases} y_n = (1-\zeta_n)P_{\mathcal{K}}(x_n) + \zeta_n T(P_{\mathcal{K}}(x_n)), \\ z_n = P_{\mathcal{E}_n}[x_n - \mu\zeta_n(y_n - T(y_n))], \end{cases}$$

where $\mathcal{E}_n = \{z \in \mathcal{E} : \langle (1-\zeta_n)P_{\mathcal{K}}(x_n) + \zeta_n T(P_{\mathcal{K}}(x_n)) - y_n, z - y_n\rangle \le 0\}$. Compute

$$\zeta_{n+1} = \begin{cases} \min\left\{\sigma, \dfrac{\mu\langle y_n - Ty_n, z_n - y_n\rangle}{\langle P_{\mathcal{K}}(x_n) - TP_{\mathcal{K}}(x_n), z_n - y_n\rangle - \left(\frac{3-2\kappa}{2-2\kappa}\right)\|P_{\mathcal{K}}(x_n) - y_n\|^2 - \left(\frac{3-2\kappa}{2-2\kappa}\right)\|z_n - y_n\|^2 + 1}\right\}, \\ \quad \textit{if} \quad \dfrac{\mu\langle y_n - Ty_n, z_n - y_n\rangle}{\langle P_{\mathcal{K}}(x_n) - TP_{\mathcal{K}}(x_n), z_n - y_n\rangle - \left(\frac{3-2\kappa}{2-2\kappa}\right)\|P_{\mathcal{K}}(x_n) - y_n\|^2 - \left(\frac{3-2\kappa}{2-2\kappa}\right)\|z_n - y_n\|^2 + 1} > 0, \\ \sigma \quad \textit{else}. \end{cases}$$

Then, the sequence $\{x_n\}$ converges strongly to an element $x^* = P_{Fix(T)}(\theta)$.

The variational inequality problem is defined in the following way:

$$\text{Find } x^* \in \mathcal{K} \text{ such that } \langle G(x^*), y - x^*\rangle \geq 0, \quad \forall y \in \mathcal{K}.$$

Note: If $f(x, y) := \langle G(x), y - x\rangle$ for all $x, y \in \mathcal{K}$, the EP transforms into variational inequality problem with $L = 2c_1 = 2c_2$ (for details see [40]). Moreover, we have

$$\begin{cases} y_n = \arg\min\limits_{y\in\mathcal{K}} \left\{\zeta_n f(x_n, y) + \frac{1}{2}\|x_n - y\|^2\right\} = P_{\mathcal{K}}[x_n - \zeta_n G(x_n)], \\ z_n = \arg\min\limits_{y\in\mathcal{E}_n} \left\{\mu\zeta_n f(y_n, y) + \frac{1}{2}\|x_n - y\|^2\right\} = P_{\mathcal{E}_n}[x_n - \mu\zeta_n G(y_n)]. \end{cases} \tag{5.1}$$

Suppose that G meets the following conditions:
(G1) G is pseudomonotone on $\mathcal{K}$ with $VI(G, \mathcal{K})$ is non-empty;
(G2) G satisfies L-Lipschitz continuity on $\mathcal{K}$ through $L > 0$;
(G3) $\limsup_{n\to\infty}\langle G(x_n), y - x_n\rangle \leq \langle G(p), y - p\rangle$ for every $y \in \mathcal{K}$ and $\{x_n\} \subset \mathcal{K}$ satisfying $x_n \rightharpoonup p$.

Corollary 5.3. *Let a mapping $G : \mathcal{K} \to \mathcal{E}$ satisfy conditions* (G1)–(G3). *Assume that sequence $\{x_n\}$ is generated as follows:*

(i) *Let $x_1 \in \mathcal{K}$, $\sigma < \min\left\{1, \frac{1}{L}\right\}$, $\mu \in (0, \sigma)$, $\zeta_1 > 0$, $\delta_n \subset [\delta, 1)$ with $0 < \delta < 1$ and $\alpha_n \subset (0, 1)$ such that*

$$\lim_{n\to\infty} \alpha_n = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \alpha_n = \infty.$$

(ii) *Compute $x_{n+1} = P_{\mathcal{K}}[\alpha_n x_n + (1 - \alpha_n)z_n - \alpha_n\delta_n x_n]$, where*

$$\begin{cases} y_n = P_{\mathcal{K}}[x_n - \zeta_n G(x_n)], \\ z_n = P_{\mathcal{E}_n}[x_n - \mu\zeta_n G(y_n)], \end{cases}$$

where $\mathcal{E}_n = \{z \in \mathcal{E} : \langle x_n - \zeta_n G(x_n) - y_n, z - y_n\rangle \leq 0\}$. Compute

$$\zeta_{n+1} = \begin{cases} \min\left\{\sigma, \dfrac{\mu\langle Gy_n, z_n - y_n\rangle}{\langle Gx_n, z_n - y_n\rangle - \frac{L}{2}\|x_n - y_n\|^2 - \frac{L}{2}\|z_n - y_n\|^2 + 1}\right\}, \\ \quad \text{if} \quad \dfrac{\mu\langle Gy_n, z_n - y_n\rangle}{\langle Gx_n, z_n - y_n\rangle - \frac{L}{2}\|x_n - y_n\|^2 - \frac{L}{2}\|z_n - y_n\|^2 + 1} > 0, \\ \sigma \quad \text{else.} \end{cases}$$

Then, the sequences $\{x_n\}$ strongly converge to a solution $x^* \in VI(G, \mathcal{K})$.

Corollary 5.4. *Let a mapping $G : \mathcal{K} \to \mathcal{E}$ satisfy conditions* (G1)–(G3). *Assume that sequence $\{x_n\}$ is generated as follows:*

(i) *Let $x_1 \in \mathcal{E}$, $\sigma < \min\left\{1, \frac{1}{L}\right\}$, $\mu \in (0, \sigma)$, $\zeta_1 > 0$, $\delta_n \subset [\delta, 1)$ with $0 < \delta < 1$ and $\alpha_n, \beta_n \subset (0, 1)$ such that*

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty \quad \text{and} \quad \liminf_{n\to\infty} \beta_n(1 - \beta_n) > 0.$$

(ii) *Compute $x_{n+1} = \alpha_n(1 - \delta_n)x_n + (1 - \alpha_n)[\beta_n z_n + (1 - \beta_n)x_n]$, where*

$$\begin{cases} y_n = P_{\mathcal{K}}[P_{\mathcal{K}}(x_n) - \zeta_n G(P_{\mathcal{K}}(x_n))], \\ z_n = P_{\mathcal{E}_n}[P_{\mathcal{K}}(x_n) - \mu\zeta_n G(y_n)], \end{cases} \tag{5.2}$$

where $\mathcal{E}_n = \{z \in \mathcal{E} : \langle P_{\mathcal{K}}(x_n) - \zeta_n G(P_{\mathcal{K}}(x_n)) - y_n, z - y_n\rangle \le 0\}$. *Compute*

$$\zeta_{n+1} = \begin{cases} \min\left\{\sigma, \dfrac{\mu\langle Gy_n, z_n - y_n\rangle}{\langle G(P_{\mathcal{K}}(x_n)), z_n - y_n\rangle - \frac{L}{2}\|P_{\mathcal{K}}(x_n) - y_n\|^2 - \frac{L}{2}\|z_n - y_n\|^2 + 1}\right\}, \\ \quad \text{if } \dfrac{\mu\langle Gy_n, z_n - y_n\rangle}{\langle G(P_{\mathcal{K}}(x_n)), z_n - y_n\rangle - \frac{L}{2}\|P_{\mathcal{K}}(x_n) - y_n\|^2 - \frac{L}{2}\|z_n - y_n\|^2 + 1} > 0, \\ \sigma \quad \text{else}. \end{cases}$$

Then, the sequence $\{x_n\}$ *strongly converges to a solution* $x^* \in VI(G, \mathcal{K})$.

6 Numerical illustration

The numerical discussion provided in this section demonstrates the efficiency of our proposed algorithms compared to Algorithms 3.1 and 3.2 in [31]. The MATLAB program was run on a PC (with Intel(R) Core(TM)i3-4010U CPU @ 1.70 GHz 1.70 GHz, RAM 4.00 GB) in MATLAB version 9.5 (R2018b). We use the built-in MATLAB fmincon function to solve minimization problems in all algorithms.

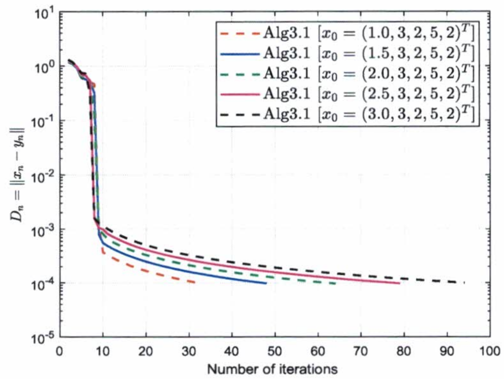
(i) The values for control parameters for Algorithm 3.1 (Alg3.1) and Algorithm 3.2 (Alg3.2) in [31] are $\alpha_n = \frac{1}{40n}$, $\delta_n = \frac{1}{5} + \frac{1}{5n}$, $\zeta_n = \frac{n}{7+2c_1 n}$, $\beta_n = \frac{1}{6} + \frac{1}{6n}$ and $D_n = \|x_n - y_n\| \le \epsilon$.

(ii) The values for control parameters for Algorithm 1 (Alg.1) and Algorithm 2 (Alg.2) are $\alpha_n = \frac{1}{40n}$, $\delta_n = \frac{1}{5} + \frac{1}{5n}$, $\beta_n = \frac{1}{6} + \frac{1}{6n}$, $\mu = \frac{5}{13c_1}$, $\zeta_1 = \frac{1}{3c_1}$ and $D_n = \|x_n - y_n\| \le \epsilon$.

Example 6.1. Assume that $f : \mathcal{K} \times \mathcal{K} \to \mathbb{R}$ is

$$f(x, y) = \sum_{i=2}^{5}(y_i - x_i)\|x\|, \quad \forall x, y \in \mathbb{R}^5,$$

where $\mathcal{K} = \{(x_1, \ldots, x_5) : x_1 \ge -1, x_i \ge 1,\ i = 2, \ldots, 5\}$. Thus, bifunction f is Lipschitz-type continuous through $c_1 = c_2 = 2$ and meet the criterion (C1)–(C4). The solution set of an EP is $EP(f, \mathcal{K}) = \{(x_1, 1, 1, 1, 1) : x_1 \ge -1\}$ (see [31]). The numerical results are shown in Figures 1–4 and Table 1 and $\epsilon = 10^{-4}$.



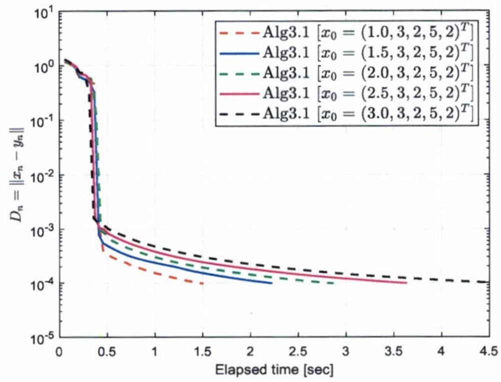


Figure 1: Example 6.1: Choosing different initial points and behavior for Algorithm 3.1 in [31].

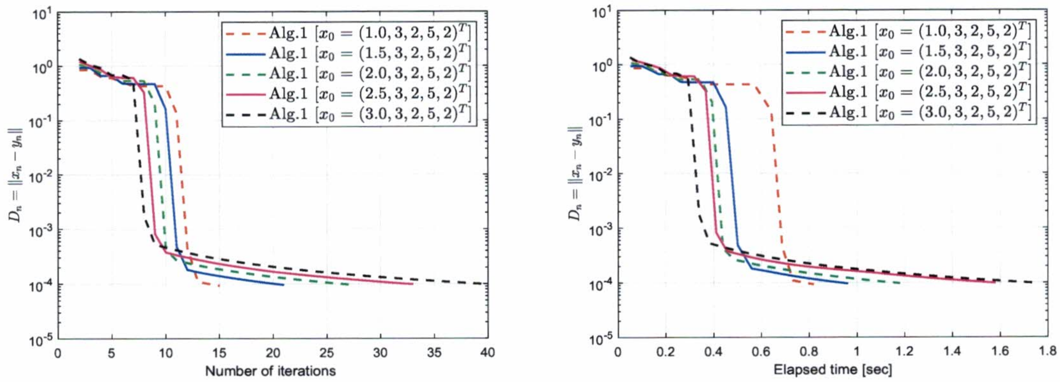


Figure 2: Example 6.1: Choosing different initial points and behavior for Algorithm 1.

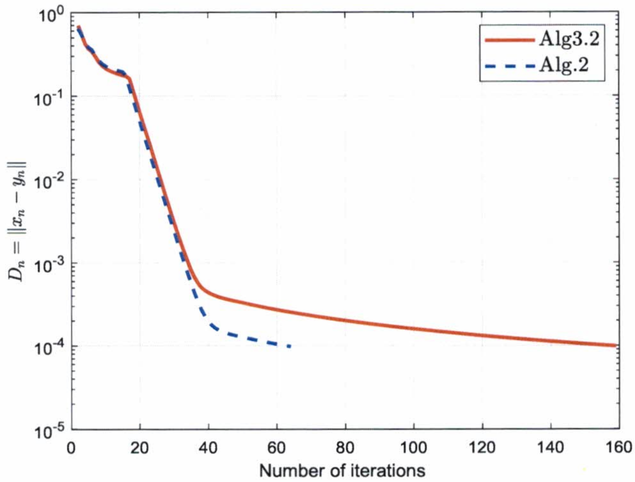


Figure 3: Example 6.1: Comparison of Algorithm 2 with Algorithm 3.2 in [31].

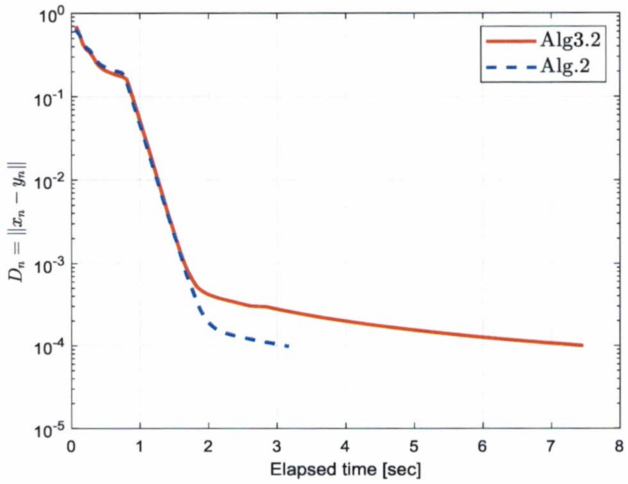


Figure 4: Example 6.1: Comparison of Algorithm 2 with Algorithm 3.2 in [31].

Table 1: Example 6.1: Numerical values of Algorithm 1 and Algorithm 3.1 in [31]

x_1	Number of iterations		CPU time in seconds	
	Alg3.1	Alg.1	Alg3.1	Alg.1
$(1.0, 3, 2, 5, 2)^T$	33	15	1.5027	0.8189
$(1.5, 3, 2, 5, 2)^T$	48	21	2.2226	0.9622
$(2.0, 3, 2, 5, 2)^T$	64	27	2.8630	1.1799
$(2.5, 3, 2, 5, 2)^T$	79	33	3.6347	1.5886
$(3.0, 3, 2, 5, 2)^T$	94	40	4.4637	1.7434

Example 6.2. Let there be n firms that assemble the same commodity. Consider that a vector x in which each element x_i remains for the quantity of the substance produces by a firm i. We see the cost function P as a decreasing affine function, which relies on the amount of $S = \sum_{i=1}^{m} x_i$, i.e., $P_i(S) = \alpha_i - \psi_i S$, where $\alpha_i > 0$, $\psi_i > 0$. The function of earnings of every firm i is defined by $F_i(x) = P_i(S)x_i - t_i(x_i)$, where $t_i(x_i)$ is the tax value and cost for developing item x_i. Consider that $\mathcal{K}_i = [x_i^{\min}, x_i^{\max}]$ is the collections of measures related to any company i, and the game plan works out for the whole design and takes the form as $\mathcal{K} := \mathcal{K}_1 \times \mathcal{K}_2 \times \cdots \times \mathcal{K}_n$. Each company wants to attain its peak earnings by pursuing the respective stage of production on the premise that the performance of the other firms is an input parameter. The commonly used modelling methodology is based on the famous Nash equilibrium principle. We would like to point out that point $x^* \in \mathcal{K} = \mathcal{K}_1 \times \mathcal{K}_2 \times \cdots \times \mathcal{K}_n$ is the point of equilibrium of the model if $F_i(x^*) \geq F_i(x^*[x_i])$, $\forall x_i \in \mathcal{K}_i$, $\forall i = 1, 2, \ldots, n$, with the vector $x^*[x_i]$ representing the vector obtained from x^* by taking x_i^* with x_i. Furthermore, let $f(x, y) := \beta(x, y) - \beta(x, x)$ with $\beta(x, y) := -\sum_{i=1}^{n} F_i(x[y_i])$, and the problem of getting the Nash equilibrium point of the model may be as follows:

$$\text{Find } x^* \in \mathcal{K} : f(x^*, y) \geq 0, \ \forall y \in \mathcal{K}.$$

It follows from the paper [29] that the bifunction f could be taken in the following form:

$$f(x, y) = \langle Ax + By + c, y - x \rangle,$$

where $c \in \mathbb{R}^5$ and A, B are

$$A = \begin{pmatrix} 3.1 & 2 & 0 & 0 & 0 \\ 2 & 3.6 & 0 & 0 & 0 \\ 0 & 0 & 3.5 & 2 & 0 \\ 0 & 0 & 2 & 3.3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix} \quad B = \begin{pmatrix} 1.6 & 1 & 0 & 0 & 0 \\ 1 & 1.6 & 0 & 0 & 0 \\ 0 & 0 & 1.5 & 1 & 0 \\ 0 & 0 & 1 & 1.5 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} \quad c = \begin{pmatrix} 1 \\ -2 \\ -1 \\ 2 \\ -1 \end{pmatrix}$$

while Lipschitz constants $c_1 = c_2 = \frac{1}{2}\|A - B\|$ (see [29]). The feasible set $\mathcal{K} \subset \mathbb{R}^5$ is

$$\mathcal{K} := \{x \in \mathbb{R}^5 : -5 \leq x_i \leq 5\}.$$

Figures 5 and 6 and Table 2 show numerical results by letting different initial values.

Example 6.3. Suppose that $\mathcal{E} = L^2([0, 1])$ is a Hilbert space with

$$\|x\| = \sqrt{\int_0^1 |x(t)|^2 dt}$$

and the inner product $\langle x, y \rangle = \int_0^1 x(t)y(t)dt$, $\forall x, y \in \mathcal{E}$. Assume that $\mathcal{K} := \{x \in L^2([0, 1]) : \|x\| \leq 1\}$ be the unit ball. Let us define an operator $G : \mathcal{K} \to \mathcal{E}$ by

$$G(x)(t) = \int_0^1 (x(t) - H(t, s)f(x(s)))ds + g(t),$$

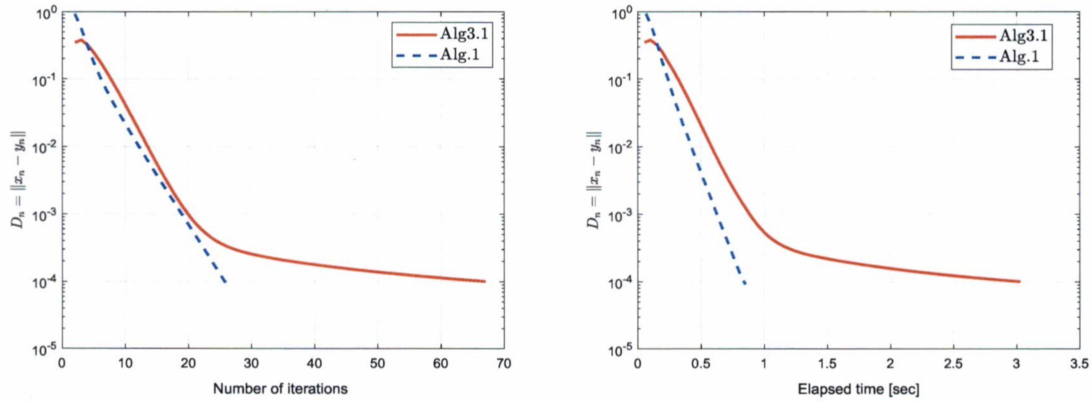


Figure 5: Example 6.2: Comparison of Algorithm 1 with Algorithm 3.1 in [31]. Comparison by using $x_1 = (1, 1, 1, 1, 1)^T$.

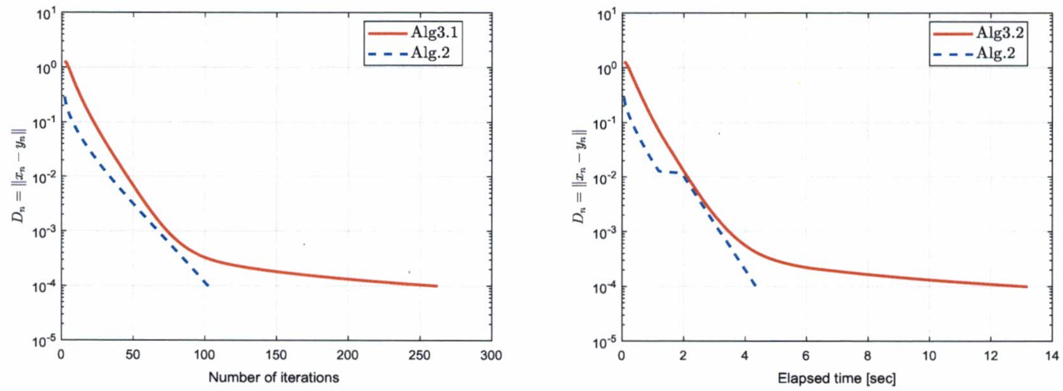


Figure 6: Example 6.2: Algorithm 2 and Algorithm 3.2 in [31]. Comparison by using $x_1 = (1, 1, 1, 1, 1)^T$.

Table 2: Example 6.2: Algorithmic comparison of Algorithms 1–2 with Algorithms 3.1–3.2 in [31]

x_1	Number of iteration				CPU time in seconds			
	Alg3.1	**Alg.1**	**Alg3.2**	**Alg.2**	**Alg3.1**	**Alg.1**	**Alg3.2**	**Alg.2**
(0, 0, 0, 0, 0)	67	28	260	108	3.0321	0.8480	13.1667	4.5632
(1, 1, 1, 1, 1)	67	26	262	103	3.0266	3.8970	13.1867	4.3772
(2, −1, −1, 2, 2)	79	32	261	103	3.2310	1.0240	13.1867	4.3512
(1, −2, 3, −4, 5)	69	30	262	101	3.4487	1.0021	13.1867	4.3123
(2, −1, 3, −4, 5)	75	37	264	108	3.8215	1.5810	13.1867	4.4987

where

$$H(t,s) = \frac{2tse^{(t+s)}}{e\sqrt{e^2-1}}, \quad f(x) = \cos x, \quad g(t) = \frac{2te^t}{e\sqrt{e^2-1}}.$$

Table 3: Example 6.3: Algorithmic comparison of Algorithms 1–2 with Algorithms 3.1–3.2 in [31]

x_1	Number of iteration				CPU time in seconds			
	Alg3.1	Alg.1	Alg3.2	Alg.2	Alg3.1	Alg.1	Alg3.2	Alg.2
t	33	5	83	11	0.005131	0.0007435	0.01423	0.001465
$2t^2$	36	6	84	15	0.005243	0.0007876	0.01612	0.001786
e^t	45	9	97	22	0.006754	0.0009867	0.01922	0.001987
$\sin(t)$	40	6	84	19	0.004987	0.0007927	0.01677	0.001699

As shown in [41], G is monotone and L-Lipschitz-continuous through $L = 2$. Table 3 shows the numerical results by taking different initial values.

7 Conclusion

We have provided two extragradient-like methods to figure out a pseudo-monotone EP that requires the Lipschitz-like condition. The algorithms use a new step-size rule that is revised on each iteration, depending on prior iterations. Strong convergence results are obtained by letting certain mild conditions on the bi-function. Many numerical experiments are presented to demonstrate the numerical behavior of the proposed methods.

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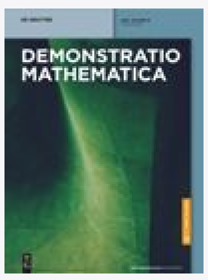

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