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กราฟเคย์เลย์ยูนิทารีของกรุปริงจำกัด

On Unitary Cayley Graphs of Finite Groupings

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สาขาวิชาคณิตศาสตร์ คณะวิทยาศาสตร์และเทคโนโลยี

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ประจำปีงบประมาณ 2564

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Abstract

A unitary Cayley graph of a groupring R_G , denoted by $\Gamma(R_G)$ is a graph with R_G as a vertex set where $\{\alpha, \beta\}$ is an edge if and only if $\alpha - \beta$ is a unit. In this research, we prove some basic properties of a unitary Cayley graph of a finite groupring such as vertices adjacent, degree of a vertex, connectivity and a dominating set.

Keywords : unitary Cayley graphs, groupings, ring

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Chapter 1

Introduction

Studying a group structure by using a graph was first studied by Arthur Cayley in 1878. Investigating on assign a group structure to the Cayley graph has been widely study by many authors. In 1988 Beck [2] studied a graph that was constructed by a commutative ring. Afterwards, the graphs associated with ring and the unitary Cayley graph has been extensively studied, such as coloring, chromatic number, clique number etc. (see [3], [8], [7], [5], [9]). However, there are some articles that investigate on a non-commutative ring.

In 2002, Milies and Sehgal [10] studied an interesting algebraic structure called a groupring, which was presented in a book, called "An Introduction to Group rings". In a final part of a brief history section on group ring of the book, the authors gave us a state of Arthur Cayley's article, "if we consider the elements of a (finite) group as basic units of a hypercomplex system", and conclude that the notion of a groupring appears for the first time.

A groupring is an algebraic structure that was combined a group and a ring by module functions. In addition, each element of a groupring consists two parts, ones a basis part from the group and a scalar part from a ring. Besides, one of the interesting points of studying a groupring structure is a groupring's structure more general than a vector space's structure.

In this research, we study a graph which is originated from a studying of Arthur Cayley, called unitary Cayley graph, associate to an algebraic structure which the notion was first appears in Arthur Cayley's article, called groupring.

The report is organized as follows: In Chapter 2, we first give some useful basic definition and properties. In Chapter 3, the methodology and literature review were presented. After that, in Chapter 4 is a main results of this study. Finally, we provide the conclusions of this research in Chapter 5.

Chapter 2

Preliminaries

2.1 Groups Rings and Modules

There are some preliminary information to study in the groupring. The definition of a group and ring are referred from [4]. Some useful notations and relevant terminologies related to this research are given in this section.

2.1.1 Groups

Definition 2.1.1 If G is a nonempty set, a **binary operation** on G is a function $*$: $G \times G \rightarrow G$.

In general, the notation $a * b$ is used for the image of (a, b) under the binary operation $*$.

Definition 2.1.2 A **semigroup** is a nonempty set G together with a binary operation $*$ on G which is

i) associative: $a * (b * c) = (a * b) * c$ for all $a, b, c \in G$;

a **monoid** is a semigroup G which contains an

ii) identity element $e \in G$ such that $a * e = e * a = a$ for all $a \in G$.

A **group** is a monoid G such that

iii) for every $a \in G$ there exists an inverse element $-a$ such that $a * (-a) = (-a) * a = e$.

A semigroup G is said to be **abelian** or **commutative** if its binary operation is

iv) commutative: $a * b = b * a$ for all $a, b \in G$.

We shall denote by $(G, *)$ a group G where $*$ is a binary operation on G .

2.1.2 Rings

Definition 2.1.3 A **ring** is a nonempty set R with two binary operations, denoted as addition $\oplus$ and multiplication $\odot$ such that: for all $a, b, c \in R$

i) $(R, \oplus)$ is an abelian group;

ii) $(a \odot b) \odot c = a \odot (b \odot c)$

iii) $a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c)$ and $(a \oplus b) \odot c = (a \odot c) \oplus (b \odot c)$

If in addition:

iv) $a \odot b = b \odot a$

then R is said to be a **commutative ring**. If R contains an element 1_R such that

v) $1_R \odot a = a \odot 1_R = a$

then R is said to be a **ring with identity** and an element 1_R is called a **unity** of R . We shall denote by $(R, \oplus, \odot)$ a ring R where $\oplus$ and $\odot$ are two binary operations on R .

Definition 2.1.4 ideal and maximal ideal

2.1.3 Modules

Definition 2.1.5 Let $(R, \oplus, \odot)$ be a ring with a unity 1_R . An abelian group $(G, *)$ together with a function $\cdot : R \times G \rightarrow G$ is called a **(left) R -module** (or a **left module over R**) if :

- i) $(a \oplus b) \cdot g = (a \cdot g) \oplus (b \cdot g),$
- ii) $a \cdot (g_1 * g_2) = (a \cdot g_1) * (a \cdot g_2),$
- iii) $a \cdot (b \cdot g) = (a \odot b) \cdot g,$
- iv) $1_R \cdot g = g,$

for all $a, b \in R$ and $g, g_1, g_2 \in G$.

For convenience, we shall refer to ag as the image of (a, g) of the above function.

2.2 Groupings

In this section, we introduce a definition of a grouping and investigate some properties of a unit on the grouping. The following definitions and some notation are given by Miles [10]. For convenience, we shall refer $e_G, 0_R$ and 1_R to an identity of a group G , an identity element of a ring R and a unity of a ring R , respectively.

2.2.1 Groupings

Let $(G, *)$ be a R -module over a ring $(R, \oplus, \odot)$ with a unity 1_R and R_G be the set of all formal linear combinations of the form

$$\alpha = \sum_{g \in G} a_g g$$

where $a_g \in R$ and $a_g = 0_R$, almost everywhere, that is, only finite number of a_g are different from 0_R where 0_R is an identity element in R and define the *support* of α to be the set of elements g of G such that the corresponding $a_g \neq 0_R$, that is

$$\text{supp}(\alpha) = \{g \in G : a_g \neq 0_R\}.$$

Given two elements $\alpha = \sum_{g \in G} a_g g$ and $\beta = \sum_{g \in G} b_g g$ in R_G , we have $\alpha = \beta$ if and only if $a_g = b_g$ for all $g \in G$. The *sum* of α and β , denoted by $\alpha + \beta$, is defined by

$$\alpha + \beta = \sum_{g \in G} a_g g + \sum_{g \in G} b_g g = \sum_{g \in G} (a_g \oplus b_g) g.$$

Also, the *product* of α and β , denoted by $\alpha\beta$, is defined by

$$\alpha\beta = \sum_{g, h \in G} (a_g \odot b_h)(g * h).$$

By reordering the terms in the formula above, we can write

$$\alpha\beta = \sum_{u \in G} c_u u$$

where $c_u = \sum_{g * h = u} a_g b_h$.

It is easy to check that, R_G is a ring under the operations above. Moreover, R_G is a commutative ring if and only if R is a commutative ring. In this research, we shall study the ring R which has no zero divisor.

Definition 2.2.1 Let $(G, *)$ be a R -module over a ring $(R, \oplus, \odot)$ with a unity 1_R . The set R_G , with the operations defined above, is called **grouping of G over R** (or **R_G is a grouping**).

2.2.2 Unity and Unit of R_G

Let X be a set, the cardinality of a set X is denoted by $|X|$. Clearly that, the cardinality of a groupring R_G is equal to m^n where $|G| = n$ and $|R| = m$. The following lemma shows some properties of $\text{supp}(\alpha)$ and $\text{supp}(\alpha\beta)$ that will be used to consider a unit element of the groupring R_G in the sequel.

Lemma 2.2.1 [4] Let R be a ring with has no zero divisor. Then $gg' = 0$ if and only if either $g = 0$ or $g' = 0$.

Lemma 2.2.2 Let $\alpha, \beta \in R_G$. The following conditions hold.

- i) If $\text{supp}(\alpha) \neq \text{supp}(\beta)$ then $\alpha \neq \beta$;
- ii) $|\text{supp}(\alpha + \beta)| \leq |\text{supp}(\alpha)| + |\text{supp}(\beta)|$;
- iii) $|\text{supp}(\alpha\beta)| \geq \max\{|\text{supp}(\alpha)|, |\text{supp}(\beta)|\}$

Proof i) It is obvious by definition.

ii) It is easy to check that, $\text{supp}(\alpha + \beta) \subseteq (\text{supp}(\alpha) \cup \text{supp}(\beta))$. Thus $|\text{supp}(\alpha + \beta)| \leq |\text{supp}(\alpha)| + |\text{supp}(\beta)|$.

iii) Since $\text{supp}(\alpha\beta) = \{gh : g \in \text{supp}(\alpha), h \in \text{supp}(\beta)\}$, we conclude that $|\text{supp}(\alpha\beta)| \geq |\{gh : g \in \text{supp}(\alpha), h \in \text{supp}(\beta)\}| \geq \max\{|\text{supp}(\alpha)|, |\text{supp}(\beta)|\}$. □

Clearly that, the converse of i) not true.

Recall that, an element u of a ring R is a unit if there exists $v \in R$ such that $uv = vu = 1_R$. Here, we investigate a unit element of a groupring R_G .

Consider, an element $1_R e_G \in R_G$, we obtain that

$$\begin{aligned} 1_R e_G \left(\sum_{g \in G} a_g g \right) &= \sum_{g \in G} (1_R \odot a_g) (e_G * g) \\ &= \sum_{g \in G} (a_g \odot 1_R) (g * e_G) \\ &= \left(\sum_{g \in G} a_g g \right) 1_R e_G \\ &= \sum_{g \in G} a_g g. \end{aligned}$$

Thus an element $u_{R_G} := 1_R e_G$ is a unity of the groupring R_G .

Lemma 2.2.3 $\alpha \in R_G$ is a unit if and only if $|\text{supp}(\alpha)| = 1$

Proof Let α be a unit. Then there exist $\alpha^{-1} \in R_G$ such that $\alpha\alpha^{-1} = u_{R_G} = 1_R e_G$. Thus $\text{supp}(\alpha\alpha^{-1}) = \text{supp}(1_R e_G) = \{e_G\}$. Since $1 = |\text{supp}(\alpha\alpha^{-1})| \geq \max\{|\text{supp}(\alpha)|, |\text{supp}(\alpha^{-1})|\}$, we can conclude that $|\text{supp}(\alpha)| = |\text{supp}(\alpha^{-1})| = 1$.

Conversely, assume that $|\text{supp}(\alpha)| = 1$. Then we put $\alpha = a_g g$ for some $a_g \in R, g \in G$. There exists $\beta = a_g^{-1}(-g) \in R_G$ such that $\alpha\beta = (a_g g)(a_g^{-1}(-g)) = a_g a_g^{-1}(g - g) = 1_R e_G$. Therefore α is a unit in R_G . □

2.3 Unitary Cayley Graphs

In this section, we start with some useful definitions in graph theory, the meaning of dominating set and a domination number of a graph. After that, the concepts of the unitary Cayley graph is offered.

2.3.1 Graphs

Definition 2.3.1 A **graph** is a pair $G = (V(G), E(G))$ of sets such that the elements of $E(G)$ are 2-element subset of $V(G)$. The elements of $V(G)$ are the **vertices** of the graph G , the elements of $E(G)$ are its **edges**.

Definition 2.3.2 Two vertices u, v of a graph G are **adjacent**, or **neighbors**, if $\{u, v\}$ is an edge of G . The set of neighbors of a vertex v in G is denoted by $N(v)$.

If a vertex $v \in e$ for some an edge $e \in E$ then v is **incident** with an edge e and the edge e is said to be **edge at** v . The **degree** $d(v)$ of a vertex v is the number of neighbor of v i.e. $d(v) = |N(v)|$.

Definition 2.3.3 Let $G = (V, E)$ be a graph. If $V' \subseteq V$ and $E' \subseteq E$, then $G' = (V', E')$ is said to be a **subgraph** of G .

If G' is a subgraph of G and G' contains all the edges $\{u, v\} \in E$ with $u, v \in V'$, then G' is an **induced subgraph** of G .

Definition 2.3.4 A **dominating set** in a graph G is a subset D of the vertex set V of G such that for every vertex in $V \setminus D$ has a neighbor in D .

A **domination number** $\gamma(G)$ of a graph G is the minimum cardinality of a minimal dominating set in G

2.3.2 Unitary Cayley Graph of Ring

Definition 2.3.5 Let R be a ring. The **unitary Cayley graph** of the ring R , is a graph with vertex set R , where $\{u, v\}$ is an edge if and only if $u - v$ is a unit of R . Denote the unitary Cayley graph of a ring R by $\Gamma(R)$.

By the definition of the edge of the unitary Cayley graph of a ring R and the form of the unit element of groupring R_G , we can conclude that the **unitary Cayley graph of the groupring** R_G , denoted by $\Gamma(R_G)$ is a graph with vertex set R_G , where $\{\alpha, \beta\}$ is an edge if and only if $\alpha - \beta$ is a unit if and only if $|supp(\alpha)| = 1$.

Chapter 3

Methodology and Literature Review

3.1 Methodology

We have 3 phases to perform this research;

- Phase I** Study background knowledge and literature reviews.
- Phase II** Verify the conditions, find some basic properties and then extend to the theorem.
- Phase III** Prove the theorems and summarize.

3.2 Literature Review

A commutative ring R is said to be local ring if R has exactly one maximal ideal.

In 2009, Akhtar et al. [1] studied a unitary Cayley graph of a finite ring R and proved that $\Gamma(R)$ is a regular graph of degree $|R^\times|$ where $R^\times$ is a set of all unit elements. Moreover, they proved that if R is a commutative local ring with maximal ideal M , then $\Gamma(R)$ is a complete multipartite graph whose partite sets are the cosets of M .

A Jacobson radical of a ring R , denoted by J_R , is defined to be the intersection of all maximal left ideal of R .

In 2012, Kiani and Aghaei [6] studied a unitary Cayley graph of a ring and prove that if $\Gamma(R) \cong \Gamma(S)$, then $\Gamma(R/J_R) \cong \Gamma(S/J_S)$ where J_R and J_S are Jacobson radicals of R and S , respectively. Moreover, they obtained that if $\Gamma(R) \cong \Gamma(R')$, then $R/J_R \cong R'/J_{R'}$ for any commutative ring R and R' .

In 2015, Su [11] submitted a doctoral thesis that studied a unitary Cayley graph of ring and many properties of the unitary Cayley graph of ring are obtained, such as girth, diameter, connectivity etc.

Chapter 4

Results

In this Chapter is about the main propose of this research which means that we shall present some properties of a unitary Cayley graph of a groupring R_G , including . Certainly, the domination number of $\Gamma(R_G)$ is offered. After that, we shall present a several interesting corollary also. On top of that, we shall decompose the unitary Cayley graph $\Gamma(R_G)$ by using a relation on the support $supp(\alpha)$ of each element α of R_G .

4.1 Unitary Cayley Graph of Ring

In this part, we classify an element of R_G by S_k such that for each $1 \leq k \leq n$, define $S_k = \{\alpha \in R_G : |supp(\alpha)| = k\}$. Note that, we let $S_0 = \{0_{R_G}\}$ and clearly that, $|S_k| = (m-1)^k$. Lastly, we prove some basic properties of the unitary Cayley graph $\Gamma(R_G)$ such as, a degree of vertex, a connectivity etc.

Lemma 4.1.1 Let $\alpha, \beta \in R_G$ where $\alpha = \sum_{g \in G} a_g g$ and $\beta = \sum_{g \in G} b_g g$. If $|supp(\alpha - \beta)| = 1$ then $|(supp(\alpha) \cup supp(\beta)) \setminus (supp(\alpha) \cap supp(\beta))| \leq 1$

Proof Let $|supp(\alpha - \beta)| = 1$. Assume to the contradiction that $|(supp(\alpha) \cup supp(\beta)) \setminus (supp(\alpha) \cap supp(\beta))| \geq 2$. Then there exists $g, g' \in (supp(\alpha) \cup supp(\beta)) \setminus (supp(\alpha) \cap supp(\beta))$ where $g \neq g'$ which means $g, g' \in (supp(\alpha) \cup supp(\beta))$ and $g, g' \notin (supp(\alpha) \cap supp(\beta))$. We consider two cases:

- i) If $g, g' \in supp(\alpha)$ then $a_g, a_{g'} \neq 0_R$. Since b_g and $b_{g'}$ are equal to 0_R , $|supp(\alpha - \beta)| \geq 2$ which is contradicts to the assumption.
- ii) If $g \in supp(\alpha)$ and $g' \in supp(\beta)$ then $a_g \neq 0_R$ and $b_{g'} \neq 0_R$. Since $g, g' \notin (supp(\alpha) \cap supp(\beta))$, $a_{g'}$ and b_g are equal to 0_R . Thus $a_g, b_{g'} \in supp(\alpha - \beta)$ this implies that $|supp(\alpha - \beta)| \geq 2$ which is contradicts to the assumption.

The case $g, g' \in supp(\beta)$ and $g' \in supp(\alpha)$, $g \in supp(\beta)$ can be proved by the similar reason. Therefore $|(supp(\alpha) \cup supp(\beta)) \setminus (supp(\alpha) \cap supp(\beta))| \leq 1$, as required. $\square$

The following theorem, we shall conclude the adjacent condition of each pair of elements in R_G .

Theorem 4.1.2 Let $\Gamma(R_G)$ be a unitary Cayley graph of a groupring R_G and $\alpha, \beta \in R_G$ where $\alpha = \sum_{g \in G} a_g g$ and $\beta = \sum_{g \in G} b_g g$. Then $\{\alpha, \beta\}$ is an edge of $\Gamma(R_G)$ if and only if the following conditions hold

- i) $|(supp(\alpha) \cup supp(\beta)) \setminus (supp(\alpha) \cap supp(\beta))| \leq 1$;
- ii) if $supp(\alpha) = supp(\beta)$ then there is a unique $g' \in supp(\beta)$ such that $b_{g'} \neq a_{g'}$ and $b_g = a_g$ for otherwise;

iii) if $|(supp(\alpha) \cup supp(\beta)) \setminus (supp(\alpha) \cap supp(\beta))| = 1$ then $b_g = a_g$ for all $g \in supp(\beta) \cap supp(\alpha)$.

Proof Let $\alpha, \beta \in R_G$ such that $\alpha = \sum_{g \in G} a_g g$ and $\beta = \sum_{g \in G} b_g g$. Assume that $\{\alpha, \beta\}$ is an edge of $\Gamma(R_G)$ which means that $\alpha - \beta$ is a unit of R_G . Thus $|supp(\alpha - \beta)| = 1$. We consider each condition as follow.

- i) Clearly that i) is obtained by Lemma 4.1.1.
- ii) Suppose that $supp(\alpha) = supp(\beta)$. Assume to the contradiction that $a_g = b_g$ for all $g \in supp(\beta) = supp(\alpha)$. Then $\alpha - \beta = \sum_{g \in G} (a_g - b_g)g = \sum_{g \in G} (a_g - a_g)g = 0_G$ this implies that $supp(\alpha - \beta) = \emptyset$ which is contradicts to $|supp(\alpha - \beta)| = 1$. Therefore the condition ii) holds.
- iii) Suppose that $|(supp(\alpha) \cup supp(\beta)) \setminus (supp(\alpha) \cap supp(\beta))| = 1$. Thus there exists $h \in (supp(\alpha) \cup supp(\beta)) \setminus (supp(\alpha) \cap supp(\beta))$. Assume to the contradiction that $b_g \neq a_g$ for some $g \in supp(\beta) \cap supp(\alpha)$. For each $g \in supp(\beta) \cap supp(\alpha)$ we obtain that $a_g - b_g \neq 0_R$ where $b_g \neq a_g$. Let $h \in (supp(\alpha) \cup supp(\beta)) \setminus (supp(\alpha) \cap supp(\beta))$. Then either $h \in supp(\alpha)$ or $h \in supp(\beta)$. Consequently, $h \in supp(\alpha - \beta)$. Choose $g \in supp(\beta) \cap supp(\alpha)$ such that $a_g \neq b_g$, then $g \in supp(\alpha - \beta)$ and clearly that $g \neq h$. Hence $h, g \in supp(\alpha - \beta)$ which is contradicts to $|supp(\alpha - \beta)| = 1$. Therefore the condition ii) holds.

Conversely, assume the conditions holds. We shall show that $|supp(\alpha - \beta)| = 1$. Since $|(supp(\alpha) \cup supp(\beta)) \setminus (supp(\alpha) \cap supp(\beta))| \leq 1$, $|(supp(\alpha) \cup supp(\beta)) \setminus (supp(\alpha) \cap supp(\beta))| \in \{0, 1\}$. Consider two cases.

- i) If $|(supp(\alpha) \cup supp(\beta)) \setminus (supp(\alpha) \cap supp(\beta))| = 0$, then $supp(\alpha) = supp(\beta)$. By using ii) we obtain that there is a unique $g' \in supp(\beta)$ such that $a_{g'} \neq b_{g'}$ and $b_g = a_g$ for otherwise. Then $a_{g'} - b_{g'} \neq 0_R$ and $a_g - b_g = 0_R$ for all $g \in supp(\beta) \setminus \{g'\}$. Consequently, $g' \in supp(\alpha - \beta)$ and $g \notin supp(\alpha - \beta)$ for all $g \in supp(\beta) \setminus \{g'\}$. Hence $supp(\alpha - \beta) = \{g'\}$. Therefore $|supp(\alpha - \beta)| = 1$ as required.
- ii) If $|(supp(\alpha) \cup supp(\beta)) \setminus (supp(\alpha) \cap supp(\beta))| = 1$. We obtain that there is $g' \in (supp(\alpha) \cup supp(\beta)) \setminus (supp(\alpha) \cap supp(\beta))$. Consequently, $g' \notin supp(\alpha) \cap supp(\beta)$ and either $g' \in supp(\alpha)$ or $g' \in supp(\beta)$. By using iii), $a_g - b_g = 0_R$ for all $g \in supp(\alpha) \cap supp(\beta)$ and $g' \in supp(\alpha - \beta)$. Hence $supp(\alpha - \beta) = \{g'\}$. Therefore $|supp(\alpha - \beta)| = 1$ which means that $\{\alpha, \beta\} \in E(\Gamma(R_G))$ as required.

□

Lemma 4.1.3 Let $\alpha \in V(\Gamma(R_G))$ and $\alpha \in S_k$. Then $N(\alpha) \subseteq S_{k-1} \cup S_k \cup S_{k+1}$.

Proof Let $\beta \in N(\alpha)$. Then $\{\alpha, \beta\} \in E(\Gamma(R_G))$. By Theorem 4.1.2 we obtain that $|(supp(\alpha) \cup supp(\beta)) \setminus (supp(\alpha) \cap supp(\beta))| \leq 1$, which means that $|supp(\alpha) \setminus supp(\beta)| \in \{0, 1\}$ and we then consider two following cases;

Case $|supp(\alpha) \setminus supp(\beta)| = 1$. Let $g \in supp(\alpha)$, $g \notin supp(\beta)$. If there exists $h \neq g$ such that $h \in supp(\beta)$ and $h \notin supp(\alpha)$ then $g, h \in (supp(\alpha) \cup supp(\beta)) \setminus (supp(\alpha) \cap supp(\beta))$ which implies that $|(supp(\alpha) \cup supp(\beta)) \setminus (supp(\alpha) \cap supp(\beta))| \geq 2$ that is a contradiction. Thus $supp(\beta) \subsetneq supp(\alpha)$. From $|supp(\alpha) \setminus supp(\beta)| = 1$ we can conclude that $\beta \in S_{k-1}$.

Case $|supp(\alpha) \setminus supp(\beta)| = 0$. We can conclude that either $supp(\alpha) = supp(\beta)$ or $supp(\alpha) \subsetneq supp(\beta)$. Clearly that, if $supp(\alpha) = supp(\beta)$ then $\beta \in S_k$. In case of $supp(\alpha) \subsetneq supp(\beta)$, we get that $|(supp(\alpha) \cup supp(\beta)) \setminus (supp(\alpha) \cap supp(\beta))| = |(supp(\alpha) \cup supp(\beta)) \setminus supp(\alpha)| \leq 1$. Since $supp(\alpha) \subsetneq supp(\beta)$, $|supp(\beta) \setminus supp(\alpha)| = 1$. Thus we can conclude that $\beta \in S_{k+1}$. $\square$

Lemma 4.1.4 Let $\alpha \in V(\Gamma(R_G))$, $\alpha \in S_k$ and $\beta \in N(\alpha)$. Then the following conditions holds;

- i) $\beta \in S_{k-1}$ if and only if $supp(\beta) \subsetneq supp(\alpha)$;
- ii) $\beta \in S_k$ if and only if $supp(\beta) = supp(\alpha)$;
- iii) $\beta \in S_{k+1}$ if and only if $supp(\alpha) \subsetneq supp(\beta)$.

Proof Let $\beta \in N(\alpha)$. Then $\{\alpha, \beta\} \in E(\Gamma(R_G))$.

- i) Let $\beta \in S_{k-1}$. Then $|supp(\alpha)| - |supp(\beta)| = 1$ which means that there exists $g \in supp(\alpha)$ such that $g \notin supp(\beta)$. Assume to the contradiction that there exists $g' \in supp(\beta)$ such that $g' \notin supp(\alpha)$. Clearly, $g \neq g'$. Thus $\{g, g'\} \subseteq (supp(\alpha) \cup supp(\beta)) \setminus (supp(\alpha) \cap supp(\beta))$ that means $|(supp(\alpha) \cup supp(\beta)) \setminus (supp(\alpha) \cap supp(\beta))| \geq 2$ which is a contradiction, since $\{\alpha, \beta\} \in E(\Gamma(R_G))$.

Conversely, Assume that $supp(\beta) \subsetneq supp(\alpha)$. Clearly that, if $|supp(\alpha) \setminus supp(\beta)| \geq 2$ then $|(supp(\alpha) \cup supp(\beta)) \setminus (supp(\alpha) \cap supp(\beta))| \geq 2$ which is a contradiction since $\beta \in N(\alpha)$. Thus $|supp(\alpha) \setminus supp(\beta)| \leq 1$ which implies that $|supp(\alpha) \setminus supp(\beta)| = 1$ because $supp(\beta) \subsetneq supp(\alpha)$. Therefore $\beta \in S_{k-1}$ as required.

- ii) Let $\beta \in S_k$. Assume to the contradiction that $supp(\beta) \neq supp(\alpha)$. Since $|supp(\alpha)| = |supp(\beta)|$, there exists $g \neq g'$ such that $g \in supp(\alpha)$, $g' \in supp(\beta)$ and $g' \notin supp(\alpha)$, $g \notin supp(\beta)$. Thus $\{g, g'\} \subseteq (supp(\alpha) \cup supp(\beta)) \setminus (supp(\alpha) \cap supp(\beta))$ that means $|(supp(\alpha) \cup supp(\beta)) \setminus (supp(\alpha) \cap supp(\beta))| \geq 2$ which is a contradiction, since $\{\alpha, \beta\} \in E(\Gamma(R_G))$.

Conversely, Let $supp(\beta) = supp(\alpha)$. It is obvious that $\beta \in S_k$, since $\alpha \in S_k$.

- iii) Let $\beta \in S_{k+1}$. Then $|supp(\beta)| - |supp(\alpha)| = 1$ which means that there exists $g' \in supp(\beta)$ such that $g' \notin supp(\alpha)$. Assume to the contradiction that there exists $g \in supp(\alpha)$ such that $g \notin supp(\beta)$. Clearly, $g \neq g'$. Thus $\{g, g'\} \subseteq (supp(\alpha) \cup supp(\beta)) \setminus (supp(\alpha) \cap supp(\beta))$ that means $|(supp(\alpha) \cup supp(\beta)) \setminus (supp(\alpha) \cap supp(\beta))| \geq 2$ which is a contradiction, since $\{\alpha, \beta\} \in E(\Gamma(R_G))$.

Conversely, Assume that $supp(\alpha) \subsetneq supp(\beta)$. Clearly that, if $|supp(\beta) \setminus supp(\alpha)| \geq 2$ then $|(supp(\alpha) \cup supp(\beta)) \setminus (supp(\alpha) \cap supp(\beta))| \geq 2$ which is a contradiction since $\beta \in N(\alpha)$. Thus $|supp(\beta) \setminus supp(\alpha)| \leq 1$ which implies that $|supp(\beta) \setminus supp(\alpha)| = 1$ because $supp(\alpha) \subsetneq supp(\beta)$. Therefore $\beta \in S_{k+1}$ as required.

□

Theorem 4.1.5 Let $\alpha \in V(\Gamma(R_G))$ and $\alpha = \sum_{g \in G} a_g g \in S_k$. Then $d(\alpha) = n(m-1)$ for all $2 \leq k \leq n-1$.

Proof By using Lemma 4.1.3, we obtain that $d(\alpha) = |N(\alpha) \cap S_{k-1}| + |N(\alpha) \cap S_k| + |N(\alpha) \cap S_{k+1}|$. Then we shall investigate the cardinality of the following 3 subsets.

- i) Consider $|N(\alpha) \cap S_{k-1}|$. By Lemma 4.1.4, we obtain that $\beta \in N(\alpha) \cap S_{k-1}$ if and only if $\{\alpha, \beta\} \in E(\Gamma(R_G))$ and $\text{supp}(\beta) \subsetneq \text{supp}(\alpha)$. By Theorem 4.1.2, we obtain that $\beta = \sum_{g \in A} a_g g$ where $A \subsetneq \text{supp}(\alpha)$ and $|A| = |\text{supp}(\alpha)| - 1$ for all $\beta \in N(\alpha) \cap S_{k-1}$. Hence $|N(\alpha) \cap S_{k-1}| = \binom{k}{k-1} = k$.
- ii) Consider $|N(\alpha) \cap S_k|$. By Lemma 4.1.4, we obtain that $\beta \in N(\alpha) \cap S_k$ if and only if $\{\alpha, \beta\} \in E(\Gamma(R_G))$ and $\text{supp}(\beta) = \text{supp}(\alpha)$. By Theorem 4.1.2, for any $\beta \in N(\alpha) \cap S_{k-1}$, we obtain that $\beta = \sum_{g \in \text{supp}(\alpha) \setminus \{h\}} a_g g + b_h h$ for some $h \in \text{supp}(\alpha)$ where $b_h \in R \setminus \{0_R, a_h\}$. Hence $|N(\alpha) \cap S_k| = \binom{k}{1}(m-2) = k(m-2)$.
- iii) Consider $|N(\alpha) \cap S_{k+1}|$. By Lemma 4.1.4, we obtain that $\beta \in N(\alpha) \cap S_{k+1}$ if and only if $\{\alpha, \beta\} \in E(\Gamma(R_G))$ and $\text{supp}(\alpha) \subsetneq \text{supp}(\beta)$. By Theorem 4.1.2, for any $\beta \in N(\alpha) \cap S_{k+1}$, we obtain that $\beta = \sum_{g \in \text{supp}(\alpha) \cup \{h\}} a_g g + b_h h$ where $\text{supp}(\alpha) \cup \{h\} = \text{supp}(\beta)$ and $b_h \neq 0_R$. Hence $|N(\alpha) \cap S_k| = \binom{n-k}{1}(m-1) = (n-k)(m-1)$.

Therefore $d(\alpha) = k + k(m-2) + (n-k)(m-1) = n(m-1)$. □

4.2 Connectivity

Let $|G| = n$, $|R| = m$ and R_G be a groupring. For each $\alpha, \beta \in R_G$ we define a relation $\sim$ by

$$\alpha \sim \beta \text{ if and only if } \text{supp}(\alpha) = \text{supp}(\beta).$$

It is easy to check that, $\sim$ is an equivalent relation on R_G ; consequently, the set of all distinct equivalence classes in $\sim$ forms a partition of R_G . By $\bar{\alpha}$ we denote the equivalence class of α i.e. $\bar{\alpha} = \{\beta \in R_G : \beta \sim \alpha\}$. Furthermore, we put $T_\alpha = \{a_g g : g \in \text{supp}(\alpha)\}$.

In this section, we investigate a connectivity of the unitary Cayley graph of groupring and obtain that $\Gamma(R_G)$ is a connected graph as follow,

Lemma 4.2.1 Let $\alpha \in V(\Gamma(R_G))$ and $\alpha = \sum_{g \in G} a_g g \in S_k$. Then an induced subgraph $\Gamma(R_G)[\bar{\alpha}]$ is a connected subgraph of $\Gamma(R_G)$.

Proof Let $\alpha = \sum_{g \in G} a_g g$, we shall show that there is a path from α to β for every $\beta \in \bar{\alpha}$. Let $\beta = \sum_{g \in G} b_g g \in \bar{\alpha}$. Then $\text{supp}(\beta) = \text{supp}(\alpha)$. Let $\{a'_{h_1} h_1, a'_{h_2} h_2, \dots, a'_{h_t} h_t\}$ be a subset of T_α such that $a'_i h_i \neq b_{h_i} h_i$ where $b_{h_i} h_i \in T_\beta$ for all $i \in \{1, 2, \dots, t\}$ and we put $T' = \{h_1, h_2, \dots, h_t\}$. Then $\alpha = \alpha_0 = \left(\sum_{g \in \text{supp}(\alpha) \setminus T'} a_g g \right) + a'_{h_1} h_1 + a'_{h_2} h_2 + \dots + a'_{h_t} h_t$. Let

$\alpha_i = \left(\sum_{g \in \text{supp}(\alpha) \setminus T'} a_g g \right) + \left(\sum_{j=1}^i b_{h_j} h_j \right) + a'_{h_{i+1}} h_{i+1} + \dots + a'_{h_t} h_t$ for all $1 \leq i \leq t$. We obtain that $\{\alpha_{i-1}, \alpha_i\} \in E(\Gamma(R_G))$ for all $1 \leq i \leq t$ and clearly that $\beta = \alpha_t$. Thus there exists a path from α to β for all $\beta \in \bar{\alpha}$ which means that for any $\beta, \beta' \in \bar{\alpha}$, there exists a path from β to α and a path from α to β' . Hence there exists a path from β to β' for all $\beta, \beta' \in \bar{\alpha}$. Therefore $\bar{\alpha}$ is a connected subgraph of $\Gamma(R_G)$ as required. $\square$

Lemma 4.2.2 Let $\alpha, \beta \in V(\Gamma(R_G))$. If $\text{supp}(\alpha) \subseteq \text{supp}(\beta)$ then α, β are connected.

Proof Let $\alpha = \sum_{g \in \text{supp}(\alpha)} a_g g$, $\beta = \sum_{g \in \text{supp}(\beta)} b_g g$ and $\text{supp}(\alpha) \subseteq \text{supp}(\beta)$. Clearly that, if $\text{supp}(\alpha) = \text{supp}(\beta)$, we get α, β are connected by Lemma 4.2.1. Suppose that $|\text{supp}(\alpha)| = k$ for some $1 \leq k \leq n$. Here, we consider the case of $\text{supp}(\alpha) \subsetneq \text{supp}(\beta)$. Let $|\text{supp}(\beta) \setminus \text{supp}(\alpha)| = t$ for some $1 \leq t \leq |\text{supp}(\beta)| - 1$. Define $\alpha_i = \sum_{g \in \text{supp}(\alpha)} a_g g + a_{h_1} h_1 + a_{h_2} h_2 + \dots + a_{h_i} h_i$ where $h_j \in \text{supp}(\beta) \setminus \left(\text{supp}(\alpha) \cup \left(\bigcup_{r=1}^{j-1} h_r \right) \right)$ for all $1 \leq j \leq i$ and $a_{h_i} \neq 0_R$ for all $1 \leq i \leq t-1$. Denote $\alpha = \alpha_0$ and we can conclude that $\{\alpha_{i-1}, \alpha_i\} \in E(\Gamma(R_G))$ for all $1 \leq i \leq t-1$. Consequently, we obtain that $\alpha_{t-1}, \beta \in E(\Gamma(R_G))$. Therefore there exists a path from α to β which means that α, β are connected as required. $\square$

Theorem 4.2.3 $\Gamma(R_G)$ is a connected graph.

Proof Let $\alpha, \beta \in V(\Gamma(R_G))$, we shall show that there exists a path from α to β . Now, let $\lambda \in V(\Gamma(R_G))$ such that $\text{supp}(\lambda) = G$. Thus $\text{supp}(\alpha) \subseteq \text{supp}(\lambda)$ and $\text{supp}(\beta) \subseteq \text{supp}(\lambda)$. By Lemma 4.2.2, we obtain that there exists a path from α to λ and a path from β to λ . Hence, we can conclude that there exist a path from α to β for all $\alpha, \beta \in V(\Gamma(R_G))$ which means that $\Gamma(R_G)$ is a connected graph as required. $\square$

4.3 A Dominating Set of $\Gamma(R_G)$

For each $\alpha \in R_G$ we denote $D(\bar{\alpha})$ a maximal pairwise non-adjacent subset of $\Gamma(R_G)[\bar{\alpha}]$.

Lemma 4.3.1 $D(\bar{\alpha})$ is a dominating set of $\Gamma(R_G)[\bar{\alpha}]$.

Proof By the definition of $D(\bar{\alpha})$ we can conclude that for any $\beta \in V(\Gamma(R_G)[\bar{\alpha}])$ there exists $\lambda \in D(\bar{\alpha})$ such that $\beta, \lambda \in E(\Gamma(R_G)[\bar{\alpha}])$. Thus $D(\bar{\alpha})$ is a dominating set of $\Gamma(R_G)[\bar{\alpha}]$ as required. $\square$

Theorem 4.3.2 $\bigcup_{\alpha \in R_G} D(\bar{\alpha})$ is a dominating set $\Gamma(R_G)$

Proof Since $D(\bar{\alpha})$ is a dominating set of the induced subgraph $\Gamma(R_G)[\bar{\alpha}]$ for all equivalence class $\bar{\alpha}$ on $V(\Gamma(R_G))$ and $\bigcup_{\alpha \in R_G} \bar{\alpha} = V(\Gamma(R_G))$, $\bigcup_{\alpha \in R_G} D(\bar{\alpha})$ is a dominating set $\Gamma(R_G)$. $\square$

Chapter 5

Conclusion

In this research, we first investigate an adjacent condition of a unitary Cayley graph of a finite groupring and obtain that

for any $\alpha, \beta \in R_G$ where $\alpha = \sum_{g \in G} a_g g$ and $\beta = \sum_{g \in G} b_g g$. We get $\{\alpha, \beta\}$ is an edge of $\Gamma(R_G)$ if and only if the following conditions hold

- i) $|(supp(\alpha) \cup supp(\beta)) \setminus (supp(\alpha) \cap supp(\beta))| \leq 1$;
- ii) if $supp(\alpha) = supp(\beta)$ then there is a unique $g' \in supp(\beta)$ such that $b_{g'} \neq a_{g'}$ and $b_g = a_g$ for otherwise;
- iii) if $|(supp(\alpha) \cup supp(\beta)) \setminus (supp(\alpha) \cap supp(\beta))| = 1$ then $b_g = a_g$ for all $g \in supp(\beta) \cap supp(\alpha)$.

Afterward, the degree of a vertex of $\Gamma(R_G)$ is obtained, i.e. $d(\alpha) = n(m-1)$ for all $\alpha \in S_k$ where $2 \leq k \leq n-1$. In addition, we prove that a unitary Cayley graph of a finite groupring $\Gamma(R_G)$ is a connected graph. Besides, we give a dominating set of $\Gamma(R_G)$ in term of union of a dominating set of each equivalence class also.

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